Mathematics: Applications and Interpretation HL

Chapter summaries

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The Mathematics: Applications and Interpretation HL course covers a wide variety of topics. As a result, this book is quite large. We understand that there is too much material in this book to work through all of the exercises from cover to cover. But we feel that it was necessary to include everything we have in order that students have the opportunity to understand the work they are doing. Teachers will need to decide which parts are essential, and which parts can be moved through more quickly, based on the abilities of their students.

CHAPTER 1: EXPONENTIALS

A Rational exponents
B Algebraic expansion and factorisation
C Exponential functions
D Graphing exponential functions from a table of values
E Graphs of exponential functions
F Exponential equations
G Growth and decay
H The natural exponential
I The logistic model

Syllabus references: SL 2.5, AHL 1.10, AHL 2.9

In the Mathematics: Applications and Interpretation HL book, we have deviated from the SL book in terms of the order in which material is presented. Exponential functions has been moved earlier so that it flows on from the function work at the end of the Mathematics: Core Topics HL book.

We begin the chapter with the study of rational exponents. This gives meaning to the value of a^x for all rational x, and lays the foundation for the study of exponential functions, which define a^x for all real x.

In the Discussion at the end of Section E, students are asked why we specify a positive base number $a \neq 1$. As a starting point, they should recognise that if $a = 1$, the graph simply becomes the horizontal line $y = 1$, which is not an exponential function. They should also find that, for negative values of a, the graph does not form a smooth curve, but instead bounces between positive and negative values. Students may find the graphing package useful for exploring functions like $y = (-2)^x$. They should realise that essentially two subgraphs are formed: the graph of $y = 2^x$, and $y = -(2^x)$. From their work on negative bases in the Mathematics: Core Topics HL book, they should understand that $y = (-2)^x$ will be positive for even integer values of x, and negative for odd integer values of x. By experimenting with fractional values of x, they may find $(-2)^x$ is defined when x is rational with an odd denominator (whether it is positive or negative depends on whether the numerator is odd or even), and undefined when x is rational with even denominator.

Since solving exponential equations algebraically is not part of the Applications and Interpretation HL course, students will use graphical methods to find the value of x for a given value of y. An Activity on equating indices is given at the end of Section F.

The work done in transforming functions in the Mathematics: Core Topics HL book will come in useful here, as students should be able to recognise that $y = 2^{-x}$ is a reflection of $y = 2^x$ in the y-axis.

In Section H, students are introduced to the natural exponential e. We present students with an Investigation to see how e appears in continuously compounding interest. Transformation of graphs again comes into play here, as students should recognise that $y = e^{rx}$ is a horizontal dilation of $y = e^x$ with scale factor $\frac{1}{r}$.

CHAPTER 2: LOGARITHMS

- **A** Logarithms in base 10
- **B** Laws of logarithms
- **C** Natural logarithms
- **D** Logarithmic functions
- **E** Logarithmic scales

Syllabus references: SL 1.5, AHL 1.9, AHL 2.10

The Opening Problem features the slide rule, which is a nice application of logarithms, and is a good basis to establish some properties of logarithms.

In Section B, we establish the laws of logarithms. We avoid expressing them in terms of a general base a, as the syllabus specifies that in exams, a will be either 10 or e. We deal with logarithms in base 10 in Sections A and B, and natural logarithms in Section C.

In Section D we consider logarithmic functions. We introduce these functions as the inverses of exponential functions. Although the syllabus only mentions logarithmic functions of the form $f(x) = a + b\ln x$, we have included functions involving horizontal translations, such as $f(x) = \ln(x-2)$, since the students already have the tools to deal with this, and it seems artificial to restrict the types of functions to be studied.

The chapter ends with the study of logarithmic scales. Logarithmic scaling can either be shown by equal values given at unequal markings, as in the slide rule, or by applying the log transformation to the values before placing them on an equally spaced scale. We have chosen to present the latter, as this is how log scaling is done in contexts the students are most likely to be familiar with, such as the Richter scale.

In terms of contextual log modelling as specified in the syllabus, we believe this is covered in the logarithmic scaling section. All of these transformations can be written in the form $a + b\ln x$. However, it is often more practical to write them in the form $n\log\left(\frac{x}{x_0}\right)$ say, since in the form $a + b\ln x$, the a has no sensible interpretation.

CHAPTER 3: APPROXIMATIONS AND ERROR

- **A** Errors in measurement
- **B** Absolute and percentage error

Syllabus references: SL 1.6

Although rounding numbers is mentioned in the syllabus, we have chosen to exclude it from the Mathematics: Applications and Interpretation HL textbook. We feel that if students are not able to round numbers at this stage, they should not be doing a HL Mathematics course. Indeed, much of the work they have already done up to this point would have required knowledge of rounding numbers.

In Example 6, we find the maximum percentage error that results from rounding. To do this, we first find the boundary values for the measurement given the rounded value. We then find the percentage error associated with each of the boundary values, and the largest value gives us the maximum percentage error. In practice, the maximum percentage error will always arise when the actual value is the lower boundary value, but we feel that this is not obviously the case, so it is more intuitive to students to try both boundary values.

CHAPTER 4: LOANS AND ANNUITIES

- **A** Loans
- **B** Annuities

Syllabus references: SL 1.7

This chapter follows on from the work done in Chapter 5 of the Mathematics: Core Topics HL book, where technology was used to solve problems involving compound interest investments. In this chapter we consider the more complex cases of loans and annuities, where regular payments or withdrawals are made from the account. It would be useful for students to see loans and annuities as the “reverse models” of each other, in which the bank and the individual are essentially swapping roles.

When using the TVM solver on the calculator, students should be made aware at this point that N is the total number of payments.

In Chapter 5 of the Core Topics HL book, we define N as "the number of compounding periods". This is the only sensible definition of N at this point, because we are dealing with investments with no regular payments or repayments. Students are instructed to give P/Y and C/Y the same value, which is the number of compounding periods per year. This allows the calculator to use N and P/Y to work out the number of years, even though there are no payments.

However, now that we are dealing with problems involving payments and repayments, students should be made aware that N is the number of payments, not the number of compounding periods. For the vast majority of problems, P/Y and C/Y will be equal, so there will not be an issue. However, in cases where P/Y and C/Y are different (for example, a person making quarterly contributions into an account which pays interest compounded monthly), students should make sure that N refers to the total number of payments.

If students are obtaining answers that differ from those in the back of the book by only a few cents, it may be due to rounding of intermediate values in calculations. For example, loan repayments are always rounded up to the next cent, rather than to the nearest cent, because if the repayment was rounded down, the loan would not be completely repaid in the specified time. The rounding used in worked examples should provide a good guide for students.

Section B contains a Discussion about superannuation, and why a government might make superannuation compulsory. In forming their answers, students should consider the consequences of a person reaching retirement age with no money saved, not only for that individual, but also for the broader society.

CHAPTER 5: MODELLING

- **A** The modelling cycle
- **B** Linear models
- **C** Piecewise models
- **D** Systems of equations

Syllabus references: SL 1.8, SL 2.5, SL 2.6

In this chapter, students are introduced to the concept of a mathematical model. We see that we can take real-life problems, and then make assumptions about the situation in order to simplify it to a form which can be represented mathematically. However, it is important to assess the reasonableness of the assumptions, and to understand that the assumptions affect the accuracy of the final answer. We must therefore strike a balance between simplicity and accuracy.

The Discussion at the end of Section A asks students to discuss the statement by George Box that "All models are wrong, but some are useful". Students should consider that any model that attempts to describe a real-life situation must involve some simplifying assumptions, and therefore is not 100% accurate. However, if the model is constructed in such a way as to correctly articulate the main characteristics of the situation, it will still be useful in making predictions about the situation it is modelling.

In Section B, we use linear models to distinguish between exact and approximate models. If the data appears to follow a linear trend, we may use a linear model to approximate the situation. The work done in this section is a precursor to what is done in Chapter 7, where we consider approximate linear models more formally.

In Section C, we introduce absolute value functions, as they are an example of piecewise linear functions. We have done this because absolute value functions will be used again in integration, when finding the area under a curve, and when considering the integral of $\frac{1}{x}$. At the end of this section, a Discussion asks students whether, given a piecewise function which changes from $f_1(x)$ to $f_2(x)$ at $x = a$, it is a requirement that $f_1(a) = f_2(a)$. Students should recognise that it depends on the situation, and in particular on the nature of the variable on the y-axis. For example, if we are modelling a continuous variable such as the height of a tree over time, the functions will need to be equal at $x = a$ as the tree's height cannot "jump" at $x = a$. However, if we are modelling a discrete variable like parking costs over time, there is no need for the functions to be equal at $x = a$, as it is perfectly reasonable for the cost to increase from \$5 to \$8 after 1 hour, for example.

In the final section, we use technology to find unknown coefficients in a model. This is done by using given information to create a 2×2 or 3×3 system of simultaneous equations. Since students should be able to solve 2×2 systems algebraically, the focus here should be on solving 3×3 systems. This method will be used in future chapters to determine quadratic, variation, and trigonometric models.

In the Research task in Section D, students must find the correct model for a free-hanging rope. They should find that the correct model is a catenary. This model is quite complex, so this is a good opportunity for students to weigh up the advantages and disadvantages of using the simpler quadratic model, or the more correct catenary model, when describing a free-hanging rope.

CHAPTER 6: DIRECT AND INVERSE VARIATION

A Direct variation
B Powers in direct variation
C Inverse variation
D Powers in inverse variation
E Determining the variation model
F Using technology to find variation models

Syllabus references: SL 2.5

The material in Sections A to D is unlikely to be assessed directly, but is important so that students understand the concept of direct and inverse proportion. That being said, this material can be progressed through quickly if the students are familiar with it.

In Section E, students are generally given the type of variation relationship that exists between two variables (for example, y is proportional to x^3), and must use given data to find the proportionality constant. The models in this section are exact.

This chapter has been placed where it is so that we complete the work on exact models, and so we are not jumping back and forth between exact and approximate models. However, this does mean that the work in Section F, where students must use technology to find the best variation model connecting two variables, is done before the students have encountered linear regression. At this stage, students should not be concerned with the correlation coefficient r. This will be explained in more detail in Chapter 7. Instead of using the correlation coefficient to assess the appropriateness of the model, students should either inspect the scatter diagram of the data, or use their knowledge of the variables involved. For example, for land blocks of fixed area, students should be able to deduce that the *width* and *length* of the block will be linked by an inverse variation model.

Students should also use the context to sensibly round the parameters of the model. For example, we may suspect a shape's *area* will be proportional to the square of its *perimeter* as it is enlarged. If the calculator returns a model with power 2.0001, this power should simply be rounded to 2.

In the Discussion in Section F, we should expect that the mass of the ball bearings is proportional to radius^3 because, assuming uniform density of the ball bearings, we would expect their mass to be proportional to their volume, which in turn is proportional to radius^3.

CHAPTER 7: BIVARIATE STATISTICS

A Association between numerical variables
B Pearson's product-moment correlation coefficient
C The coefficient of determination
D Line of best fit by eye
E The least squares regression line
F Statistical reliability and validity
G Spearman's rank correlation coefficient

Syllabus references: SL 4.4, SL 4.10, AHL 4.13

Most of this chapter is similar in style and presentation as in previous books and the MYP series.

In Section C, we introduce the coefficient of determination, r^2. Here, we tell the student that r^2 is the proportion of the variation in the dependent variable that is *explained* by the independent variable. We defer the explanation (and proof) of this until Section E (The least squares regression line), since we need the definition of SS_{res} and the idea of a fitted model before we can explain $r^2 = 1 - \dfrac{SS_{\text{res}}}{SS_{\text{tot}}}$.

In Section E, there is a Discussion regarding finding the line which minimises the horizontal, rather than vertical, distances between the data points and the line. Students should conclude that this line is different from the least squares regression line, and is found by swapping the variables used in the regression. In other words, we find the regression line of x against y, rather than y against x. This line would be more reliable for estimating x given a value of y.

Section G (Spearman's rank correlation coefficient) is new. We motivate the need for the rank correlation by focussing on the *direction* of a trend that may not necessarily be linear.

CHAPTER 8: NON-LINEAR MODELLING

A Logarithmic models
B Exponential models
C Power models
D Problem solving
E Non-linear regression

Syllabus references: SL 2.5, AHL 2.9, AHL 4.13

The main theme of this chapter is to *fit* non-linear models to data. For the first 4 sections, this is done via logarithmic transformations to obtain a linear model, on which the linear regression techniques covered in Chapter 7 can be applied.

Section E (Non-linear regression) however is completely different. At the start of the section, we recall the motivation for least squares linear regression and apply it to a distinctly non-linear model. However, unlike linear regression, we cannot easily find the regression coefficients analytically.

In general, **non-linear optimisation in multiple variables** is difficult, and numerical methods are often required to explore non-linear regression fully. Such numerical methods are far beyond the scope of this course.

Although we have provided spreadsheets and calculator instructions to do non-linear regression with technology, it should be noted that:

- The algorithm used in the example spreadsheet is *very* inefficient, and does not generalise well (in the sense that it is difficult to adapt the spreadsheet for different situations).
- Most graphics calculators **do not** actually do non-linear regression formally. The non-linear model fitting functions either use transformations (exponential, power, and logarithmic) or multiple linear regression internally to calculate coefficients (quadratic, cubic, and quartic).

Throughout the section, we have noted the limitations of the technology which would be available to the student. We believe non-linear regression is better explored with the aid of a more powerful programming language, such as Python or R.

Even if the students are not familiar with programming, discussing the approach to non-linear regression and the limitations is still worthwhile.

CHAPTER 9: VECTORS

A Vectors and scalars
B Geometric operations with vectors
C Vectors in the plane
D The magnitude of a vector
E Operations with plane vectors
F Vectors in space
G Operations with vectors in space
H The vector between two points
I Parallelism
J The scalar product of two vectors
K The angle between two vectors
L The vector product of two vectors
M Vector components

Syllabus references: AHL 3.10, AHL 3.13

This chapter is quite long, and we have described vectors from scratch as we cannot guarantee that students have seen vectors before. However, if your students have seen vectors before, it may be sensible to move quickly through the first few sections of this chapter.

In the Discussion at the end of Section A, students should conclude that a vector of length 0 does not have direction, and such a vector can only be represented geometrically as a point, rather than a line segment.

The intention of the Discussion at the end of Section B is to get students to think about the limitations of representing vectors geometrically. This is a good lead-in to the following section, which introduces vectors in component form.

Once we consider 3-dimensional vectors in Section F, it is no longer practical to represent the vectors geometrically, and so almost all of the work is done in component form. Students should see how the properties of 3-dimensional vectors extend readily from those of 2-dimensional vectors. From Section H onwards, 2-dimensional and 3-dimensional vectors are presented together.

Care should be taken to distinguish between the modulus of a scalar, and the length of a vector.

In Section L, to help students remember the formula for the vector product, we have included an Investigation which defines the determinant of a matrix, and explores how the vector product formula can be written in terms of determinants.

In Section M, we consider vector components. We find it strange that the syllabus focuses on scalar components in particular directions rather than actual vectors, but as this is what is required by the syllabus, it is how we have presented it. It may help students to think about this as an alternative to writing a vector in regular component form. When we write a vector in regular component form, we describe its components in the x-direction and the y-direction. In this case, we write a vector **a** in terms of its component in the direction of **b**, and its component perpendicular to **b**.

The only major difference is that the component perpendicular to **b** is always positive. This is addressed in the Discussion. Students should find that $\sin\theta$ is always positive for acute or obtuse angles, so the component of **a** perpendicular to **b** is always positive, so it is not signed. This is reasonable, since although **b** has a particular direction, there are infinitely many vector directions perpendicular to **b**, so there is no reason that this component should be positive in some cases and negative in others.

CHAPTER 10: VECTOR APPLICATIONS

- **A** Lines in 2 and 3 dimensions
- **B** The angle between two lines
- **C** Constant velocity problems
- **D** The shortest distance from a point to a line
- **E** The shortest distance between two objects
- **F** Intersecting lines

Syllabus references: AHL 3.11, AHL 3.12

In this chapter we use vectors to explore lines, planes, and the relationship between them.

A key difference between lines and vectors that students should keep in mind is that lines do not have inherent "direction" to them. This has important consequences when considering the uniqueness of the equation of a line, and the angle between two lines.

In the discussion in Section A, students should find that it does not make sense to talk about the gradient of a line in space.

In Section D, students must find the shortest distance from a point to a line. Students should recognise that they can use the component of one vector perpendicular to another, as studied in Chapter 9, to find this distance. However, this is not always efficient, especially if the question also requires finding the position at which this shortest distance occurs.

In Section F, students must solve equations simultaneously to find the intersection of two lines. This is done using technology. When studying 3-dimensional lines, if the system has no solutions, the lines are either parallel or skew. To determine this, students must examine the direction vectors of the lines. If the direction vectors are scalar multiples of each other, the lines are parallel, otherwise the lines are skew.

Variable velocity in two dimensions will be presented in the Kinematics chapter. This will allow students to transition between position, velocity, and acceleration. This will be especially important for understanding projectile motion.

CHAPTER 11: COMPLEX NUMBERS

A Real quadratics with $\Delta < 0$
B Complex numbers
C Operations with complex numbers
D Equality of complex numbers
E The complex plane
F Modulus and argument
G Geometry in the complex plane
H Polar form
I Exponential form
J Frequency and phase

Syllabus references: AHL 1.12, AHL 1.13

The Opening Problem and Historical Note at the start of the chapter aim to lead students to the idea that it is *useful* to define the square root of a negative number. Students should see that, once we define $i = \sqrt{-1}$, we can find the solutions to *any* quadratic equation.

It is important that students recognise the analogy between using radical conjugates to perform divisions with radicals, and using complex conjugates to perform divisions with complex numbers.

In the Discussion in Section H, students should find that defining the modulus to be positive, and defining the argument to be between $-\pi$ and π, helps ensure that the polar form for any complex number is unique.

Students should understand that writing numbers in polar form makes multiplying complex numbers much easier and gives it a geometric interpretation. This can be contrasted with the multiplication of complex numbers in Cartesian form at the end of Section E, where there appears to be no geometric connection between z, w, and zw.

In Sections H and I, students must use technology to convert complex numbers between Cartesian, polar, and exponential form. Teachers should be aware that different calculator models represent complex numbers in different ways.

In the final section, we use the exponential form to write the sum of trigonometric functions as a single trigonometric function. Teachers should use this opportunity to highlight the usefulness of the exponential form. For example, it is easier to manipulate $e^{(2t+5)i}$ than $\sin(2t + 5)$, as we can apply the usual exponent laws.

Again, teachers should be aware of the capabilities of the calculator the class is using, as this will affect how these problems are solved. For example, in Example 23, some calculator models will require converting $10e^{i} + 7e^{3i}$ to the Cartesian form $-1.53 + 9.40i$, before it is converted into a single exponential form $9.53e^{1.73i}$, whereas other models will be able to skip the intermediate step.

CHAPTER 12: MATRICES

A Matrix structure
B Matrix equality
C Addition and subtraction
D Scalar multiplication
E Matrix algebra
F Matrix multiplication
G The inverse of a matrix
H Simultaneous linear equations

Syllabus references: AHL 1.14

We begin our study of matrices by looking at the structure of matrices, and exploring how to operate with them. Many students struggle with matrix multiplication the first time they encounter it, so we first ask students to multiply row matrices by column matrices. This is a good way to introduce multiplication of larger matrices, as the element in row i, column j of **AB** is found by multiplying the ith row of **A** by the jth column of **B**.

In Section G, there is an Investigation about the properties of matrix determinants, in which it is discovered that $\det(\mathbf{AB}) = \det \mathbf{A} \times \det \mathbf{B}$. Although this property is not widely used in this chapter, it is important in Chapter 14, in the study of composite transformations. This property makes it easier to understand that, for example, a reflection followed by a rotation must be another reflection, as the determinant of the transformation matrix is $1 \times -1 = -1$.

CHAPTER 13: EIGENVALUES AND EIGENVECTORS

- **A** Eigenvalues and eigenvectors
- **B** Matrix diagonalisation
- **C** Matrix powers
- **D** Markov chains

Syllabus references: AHL 1.15, AHL 4.19

In this chapter we extend our study of matrices to consider eigenvalues and eigenvectors.

Students may initially have trouble finding the eigenvector corresponding to a particular eigenvalue, as they have not previously used a free variable t to describe an infinite set of solutions. It may help students to give an analogy to the vector equation of a line, where we used a parameter t to describe the infinite set of points on the line.

In Section B, we use eigenvalues and eigenvectors to diagonalise a square matrix. The application of this is presented in Section C, where diagonalisation is used to find powers of a matrix.

In the Discussion at the end of Section C, students should find that matrices only have a "square root" if none of its eigenvalues are negative. They should also find $\mathbf{B} = -\mathbf{A}^{\frac{1}{2}}$ is another solution to $\mathbf{B}^2 = \mathbf{A}$.

Markov chains appear in this chapter since we can use eigenvalues and eigenvectors of the transition matrix to find the steady state.

Extreme care needs to be taken with the transition matrices in this chapter. The syllabus has specified that the columns should represent the current state, and the rows should represent the future state. Presumably this is done to allow the state matrices to be written in columns, so that it can be tied into the work with eigenvectors. However, we believe this to be a serious error, in that it is unconventional and unintuitive, and indeed it is inconsistent with the transition matrices used in graph theory.

Once classes have completed Markov chains, there is definitely an option to go straight to the transition matrices for graphs in Chapter 15. After all, the transition diagrams presented here are essentially nothing more than directed graphs. However students should be made aware of the inconsistencies between the transition matrices discussed above.

CHAPTER 14: AFFINE TRANSFORMATIONS

- **A** Translations
- **B** Rotations about the origin
- **C** Reflections
- **D** Stretches
- **E** Enlargements
- **F** Composite transformations
- **G** Area

Syllabus references: AHL 3.9

We use the term affine transformation to describe the general transformations of the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ in this chapter, which includes translations with vector **b**. However, we have also defined a linear transformation as one simply of the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$, which excludes translations. This allows us to talk about a transformation matrix **A** associated with reflections, rotations, stretches, and enlargements.

We start the chapter with translations, rotations, and reflections, since these transformations do not affect the area of the object. It is unfortunate that these students are not familiar with the compound angle formulae, as it would have been useful in establishing the transition matrices for rotations and reflections. Instead we have used the exponential form of complex numbers to establish these results. While students are not required to establish these proofs, we feel it will aid in their understanding, and it is another example of the utility of the exponential form.

Without the double angle formulae, it is also harder for students to calculate the transformation matrices for reflections. Given the gradient of the reflection line $m = \tan\alpha$, students find α using the inverse $\tan$ ratio, and then calculate $\cos 2\alpha$ and $\sin 2\alpha$ using technology.

When considering composite transformations in Section F, we first deal with composite linear transformations. This allows us to say that a linear transformation with transformation matrix **A**, followed by a linear transformation with transformation matrix **B**, is itself a linear transformation with transformation matrix **BA**. We then describe an affine transformation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ as the composition of a linear transformation with transformation matrix **A**, followed by a translation through **b**.

In the Discussion at the end of Section G, students are asked if the shape of an object is always preserved under a linear transformation. Students should conclude that the shape of an object is not always preserved, for example if a circle is horizontally stretched, the resulting image is an ellipse.

There are a lot of really interesting Investigations and Activities in this chapter, and, if time permits, students should be encouraged to explore them. We would particularly recommend students have a go at the fractal geometry Activity just before the Review Sets.

CHAPTER 15: GRAPH THEORY

- **A** Graphs
- **B** Properties of graphs
- **C** Routes on graphs
- **D** Adjacency matrices
- **E** Transition matrices for graphs
- **F** Trees
- **G** Minimum spanning trees
- **H** Eulerian graphs
- **I** The Chinese Postman Problem
- **J** Hamiltonian graphs
- **K** The Travelling Salesman Problem

Syllabus references: AHL 3.14, AHL 3.15, AHL 3.16

One of the biggest issues for students first learning graph theory is the amount of terminology that needs to be learnt. The first three sections of this chapter are therefore dedicated to help students familiarise themselves with terminology. Section A largely deals with terminology for the individual parts of a graph (such as vertex, edge, loop), whereas Section B deals with terminology for describing graphs as a whole (such as simple, connected, complete). Section C deals with terminology to describe routes along a graph (such as path, trail, cycle). Familiarity with this terminology is very important when studying the Chinese Postman Problem and the Travelling Salesman Problem later in the chapter.

In Section D, we use adjacency matrices and their powers to find the number of routes between particular vertices. In a Discussion, students are asked what $\mathbf{A}^2$ tells us about the number of *paths* between vertices. Students should conclude that $\mathbf{A}^2$ does not give us any information about the number of paths, as the route may contain loops, so a 2-step route from P to Q may have the form $\text{P} \to \text{Q} \to \text{Q}$, which is not a path. More generally, the matrix $\mathbf{A}^n$ does not tell us how many n-step routes between vertices are paths. Depending on the context, this may be important as routes which backtrack to vertices we have already visited are likely to be inefficient and undesirable.

In Section E, we look at transition matrices for graphs. Students will need to take care here, as the transition matrices for graphs are the transpose of those for Markov chains. We feel that the transition matrices in this chapter are more conventional and intuitive, but we have presented the transition matrices for each chapter as is required by the syllabus. A Discussion has been included in this section to make students aware of this. This does mean that the state matrix must now be written as a row matrix, and the transition matrix must be premultiplied by the state matrix.

In Section I, students explore the Chinese Postman Problem. In solving this problem, they must find the shortest way to connect pairs of vertices of odd degree. The syllabus specifies that there should be no more than 4 vertices of odd degree, so students should be able to find the shortest way to connect these vertices by inspection. Students should also be aware that, although the minimum weight is unique, there are likely to be many possible routes that can be taken with this weight. This can be seen in that, for a given route with the minimum weight, we can immediately find another route simply by reversing the direction. The route given in the answers is only an example.

In Section K, students explore the Travelling Salesman Problem. Although the solution to this problem can be found with smaller graphs by inspection, the focus here is to use the algorithms to find upper and lower bounds for the solution.

Students should understand the difference between the algorithms for the upper and lower bounds. When we find an upper bound, we are finding a Hamiltonian cycle for the graph, which may not necessarily be of minimum weight. The process of finding a lower bound is more abstract, and students should focus on the actual weight obtained from the algorithm, rather than the edges themselves, which will most likely not form a cycle, and are relatively meaningless.

The syllabus suggests completing a table of least distances to convert a practical TSP into a classical TSP. From the syllabus and the specimen exams, it is unclear whether this will arise from graphs being non-Euclidean as well as incomplete, but we have included a couple of questions dealing with non-Euclidean graphs to make sure we have covered all possible cases.

CHAPTER 16: VORONOI DIAGRAMS

A Voronoi diagrams
B Constructing Voronoi diagrams
C Adding a site to a Voronoi diagram
D Nearest neighbour interpolation
E The Largest Empty Circle problem

Syllabus references: SL 3.6

Voronoi diagrams is a topic which is likely to be unfamiliar to many teachers and students.

We begin by presenting students with Voronoi diagrams, and asking students to interpret the diagrams. We hope this will allow students to get used to the concept and the terminology, as well as to develop an intuition as to how the diagrams should look, before the students are asked to construct a Voronoi diagram of their own.

We then guide students through the construction of Voronoi diagrams, including adding an extra site to a diagram, finding a missing site or edge, and finding the largest empty circle within a diagram.

At the end of Section D, students are asked to discuss the advantages and disadvantages of nearest neighbour interpolation. In their discussion, students should consider its ease of use, and its accuracy. Students should also note that the context in which it is used is an important consideration. In some situations, it is more important to get a reasonable estimate quickly, and in other situations, accuracy is more important. When discussing more accurate ways to perform the interpolation, students should note that the nearest neighbour algorithm does not take into account the relative distance of a point from the other sites other than its nearest site. For example, if a point is closest to site B, but is almost as close to site A, the algorithm should take this into account, and should consider the value at site A as well as at site B.

We hope that the contextual nature of these problems will make this chapter engaging for students.

CHAPTER 17: INTRODUCTION TO DIFFERENTIAL CALCULUS

A Rates of change
B Instantaneous rates of change
C Limits
D The gradient of a tangent
E The derivative function
F Differentiation from first principles

Syllabus references: SL 5.1, SL 5.3

This chapter provides students with their first look at differential calculus. Although much of the calculus content is common between the HL courses, we expect the classes will be separated by the time they encounter calculus. Having the calculus chapters in the separate books allows a more targeted approach to calculus for each course.

This chapter begins with rates of change, which is used to motivate an informal study of limits. A blended learning investigation, which combines limits, previous results, and technology, is used to explore the instantaneous rate of change for a curve.

The Discussion at the end of Section C.1 invites students to ponder the existence of limits. To answer this, it may help students to return to how we define the limit of a function. Is there any guarantee that $f(x)$ will get closer and closer to a particular value as x gets closer and closer to a? When considering the graph of $f(x) = \frac{1}{x}$, students should find that the graph approaches $-\infty$ as x approaches 0 from the left, and the graph approaches ∞ as x approaches 0 from the right. Since the graph does not approach a particular value as x approaches 0, $f(x) = \frac{1}{x}$ does not have a limit as x approaches 0.

Although differentiation from first principles is not explicitly in the syllabus, we feel that it is essential for understanding the process of differentiating a function, and there is no point in studying limits if we do not consider first principles.

At the end of Section F, there is a Discussion which asks whether a function always has a derivative function, and whether the domains of a function and its derivative are always the same. To help guide students, they should be encouraged to consider the definition of "function" as broadly as possible, for example, a collection of disconnected points is considered to be a function, as long as no pair of points share the same x-coordinate. A function does not need an "equation" which defines its set of points. Students should also consider functions such as $f(x) = \sqrt{x}$, and discuss whether this function has a limit as x approaches 0, given that the function is undefined for $x < 0$.

CHAPTER 18: RULES OF DIFFERENTIATION

A Simple rules of differentiation
B The chain rule
C The product rule
D The quotient rule
E Derivatives of exponential functions
F Derivatives of logarithmic functions
G Derivatives of trigonometric functions
H Second derivatives

Syllabus references: SL 5.3, AHL 5.9, AHL 5.10

In this chapter students will discover rules for differentiating functions. We use limits at infinity to motivate the natural exponential function e^x as the function which is its own derivative. Many of the proofs of these rules, while not required, are included to aid in students' understanding.

In Section F, we find the derivative of $y = \ln x$. Students can click on a link to obtain a graphical proof for this rule.

There are a large number of differentiation rules in this chapter. Students should not get too bogged down trying to remember the derivatives involving compositions, such as $e^{f(x)}$ or $\sin(f(x))$. If students understand how the chain rule works, then finding the derivative of $\sin(f(x))$ is straightforward as long as you can find the derivative of $\sin x$.

Throughout this chapter, students should be encouraged to remember what a derivative function means, rather than just performing the differentiation without giving thought to its meaning. To this end, we have included exercises asking students to answer questions involving the gradients of tangents to the functions they are differentiating.

CHAPTER 19: PROPERTIES OF CURVES

A Tangents
B Normals
C Increasing and decreasing
D Stationary points
E Shape
F Inflection points

Syllabus references: SL 5.2, SL 5.4, SL 5.6

This chapter allows students to apply the calculus they have learnt to discover the properties of curves.

In the previous Mathematics HL book, we dealt with tangents and normals together in a single section, however we have placed them in separate sections here as the tangents section is quite large, and there are sufficient concepts presented in this exercise to warrant a section of its own.

Students should be encouraged to think of the concepts of increasing and decreasing in terms of intervals, rather than at a particular point. This will help students understand why, for example, the graph of $y = x^2$ is increasing for $x \geqslant 0$, and decreasing for $x \leqslant 0$.

In Section D we return to sign diagrams, and see how the sign diagram of $f'(x)$ can be used to determine the nature of the stationary points of $f(x)$. As outlined in the syllabus, instructions are given to use technology to sketch the graph of $f'(x)$, which can then be used to solve $f'(x) = 0$. However, students should be encouraged to use an algebraic approach to find stationary points where possible, especially since we can use technology to find local maxima and minima directly from the graph of $f(x)$.

We have added a section dealing with shape before we introduce inflection points. This is similar to how we talk about increasing and decreasing functions before we deal with stationary points.

CHAPTER 20: APPLICATIONS OF DIFFERENTIATION

- **A** Rates of change
- **B** Optimisation
- **C** Modelling with calculus
- **D** Related rates

Syllabus references: SL 2.5, SL 5.1, SL 5.7, AHL 5.9, AHL 5.10

In this final chapter of differential calculus, we explore some of its real world applications. The important skill in this chapter is to take the calculus techniques learnt in previous chapters, apply them to real life problems, and interpret the results in the context of that problem.

Students should keep in mind the constraints imposed by the context of the problem, and to make sure their solution makes sense in this context. This is especially true in problems involving trigonometry, in which we must remember that, for example, $\sin x$ can only take values from -1 to 1.

In Section C, we use known information to find unknown coefficients in models, like we have done in previous chapters. Now, however, some of the information is given in the context of the derivative, rather than of the model itself. In particular, this section gives students an opportunity to model with cubic functions. This section ends with an online Activity about cubic spline interpolation. This could potentially be a useful starting point for a student's Mathematical Exploration.

Related rates may be a challenge for some students, especially since they have not done implicit differentiation. A good approach to explain the change in mindset for these questions is that previously we have taken a variable y in terms of x, and simply differentiated by x. Now, because we are interested in how both of these variables interact over time, we differentiate the whole equation with respect to time t, so both variables are differentiated with respect to t.

CHAPTER 21: INTRODUCTION TO INTEGRATION

- **A** Approximating the area under a curve
- **B** The Riemann integral
- **C** Antidifferentiation
- **D** The Fundamental Theorem of Calculus

Syllabus references: SL 5.5, SL 5.8

We begin our study of integration with some numerical methods for finding the area under a curve, which is consistent with the history of integration, and so ties in well with IB thinking. Limits at infinity are again used here to explore what happens as we consider more and more upper and lower rectangles.

At the end of Section A.1, there is an interesting Investigation which establishes an exact formula for the area under $y = x^2$. Students who have finished the exercises early should be encouraged to explore this Investigation. Students should recognise that here we are finding an exact formula analytically, as opposed to approximating the area numerically as in the exercises.

Section B is entitled "The Riemann Integral", which may sound complicated, but all we are really doing is giving some integral notation to describe the area under a curve. It will be beneficial for students to understand this notation before moving on to see how the area under a curve relates to the antiderivative of a function.

We then move on to consider antiderivatives of functions, culminating in the Fundamental Theorem of Calculus, which links antiderivatives and the area under a curve.

This is a short chapter, but is quite involved conceptually, so it is important that students spend the time to understand the link between antiderivatives and the area under a curve.

CHAPTER 22: TECHNIQUES FOR INTEGRATION

- **A** Discovering integrals
- **B** Rules for integration
- **C** Particular values
- **D** Integrating $f(ax+b)$
- **E** Integration by substitution

Syllabus references: SL 5.5, AHL 5.11

Now that we have established that integration is the reverse process of differentiation, we use the rules for differentiation in reverse to develop the rules for integration. At this stage we only consider indefinite integration.

As with the rules of differentiation, there are many rules for integration in this chapter. Students should be familiar with the contents of the formula booklet, and recognise that these rules can be derived from differentiation, rather than trying to memorise them all.

At the end of Section B, students are asked to discuss why we specify that our rule for integrating x^n is not valid for $n=-1$. Students should conclude that substituting $n=-1$ into this rule would result in a division by zero. They should also note that the case $n=-1$ is considered separately, and the integral of $\frac{1}{x}$ is $\ln|x|+c$.

Some students may find integration by substitution challenging, as it is not always obvious what substitution should be made. Students should find that, with practice, they will develop an intuition for the substitution which should be made.

In some instances, extra information about the original function is included, allowing us to determine the constant of integration. Occasionally simultaneous equations will be required to find multiple unknowns.

CHAPTER 23: DEFINITE INTEGRALS

- **A** Definite integrals
- **B** Definite integrals involving substitution
- **C** The area under a curve
- **D** The area above a curve
- **E** The area between a curve and the y-axis
- **F** Solids of revolution
- **G** Problem solving by integration

Syllabus references: SL 5.5, AHL 5.12

Once we have established the rules for integration, we now have more tools to calculate definite integrals, and to explore the relationship between definite integrals and areas.

When the integration requires substitution, students must make sure to transform the endpoints in the definite integral as well.

Areas under and above curves are treated separately, giving students more of an opportunity to see that the definite integral is a *signed* area function.

The syllabus specifies that some definite integrals will not be able to be performed analytically, and so technology must be used. To this end, we have included calculator instructions, screenshots, and exercises which allow students to practise finding definite integrals using technology.

In Section E, we have given the formula for the area between a curve and the y-axis as $\int_c^d f^{-1}(y)\ dy$ rather than $\int_c^d x\ dy$ to better indicate what is required for the integration, and to emphasise that the function f must be invertible. In the Discussion at the end of the section, students should conclude that f must be invertible so that we can write x as a function of y. To find the shaded area shown in the Discussion, students should think about the area under the curve $y=f(x)$, as well as the area bd of the large rectangle, and the area ac of the small rectangle.

Section G ends with an online Activity about Buffon's needle problem. Some of the integration involved, particularly in the long needle case, is quite difficult, so it should not cause students concern if they cannot do this. However, it is hoped that the more able student will find this Activity interesting and engaging.

CHAPTER 24: KINEMATICS

- **A** Displacement
- **B** Velocity
- **C** Acceleration
- **D** Speed
- **E** Velocity and acceleration in terms of displacement
- **F** Motion with variable velocity
- **G** Projectile motion

Syllabus references: AHL 5.13, AHL 3.12

Having dealt with both differentiation and integration, we will now bring these processes together in the study of kinematics. Rather than dealing first with differentiation, then with integration, we will deal with particular concepts of kinematics, and present the differentiation and integration aspects of each concept together.

Most students are likely to have previously encountered questions concerning motion in the form of travel graphs. A difficulty students are likely to encounter in this chapter is talking about displacement rather than distance, and velocity rather than speed. For this reason, we have included a brief outline of the language of motion at the start of the chapter.

In Example 4 part **b**, we are given a velocity function, and asked to find the distance travelled in the first 4 seconds. In answering this question, we choose the initial displacement to be zero. We do this because an initial displacement is not given in the question, and the choice of initial displacement of the particle does not affect its distance travelled. Setting an initial displacement allows us to perform the distance calculations without involving a constant of integration c.

In Section D, a Discussion asks students to explain why the "sign test" for speed works, by considering various scenarios. The students should use the fact that the speed is the magnitude of the velocity. So, for example, if the velocity and acceleration are both negative, this means that the velocity is negative, and its value is decreasing, which means its *magnitude* is increasing, and so its speed is increasing.

In Section E, we consider the velocity and acceleration of an object expressed as a function of its displacement, rather than of time. This is likely to be quite a conceptual shift for many students. To help understand this, students have the opportunity to discuss the motion of an object with velocity expressed in terms of its displacement.

When $v = s$, students should find that:

- for initial displacement $s = 2$, the object moves to the right indefinitely with increasing speed
- for initial displacement $s = -2$, the object moves to the left indefinitely with increasing speed
- for initial displacement $s = 0$, the object remains at $s = 0$ indefinitely.

When $v = 5 - s$, students should find that:

- for initial displacement $s = 5$, the velocity is 0, so the object remains at $s = 5$ indefinitely
- for initial displacement $s = 0$, the object moves to the right with decreasing speed, getting closer and closer to $s = 5$
- for initial displacement $s = 8$, the object moves to the left with decreasing speed, getting closer and closer to $s = 5$.

In the Discussion at the end of Section E, students should find that the formula $a = v\,\dfrac{dv}{ds}$ is only applicable when an object's velocity can be expressed as a function of its displacement. In the case of the projectile, this is clearly not the case as a projectile may obtain a particular displacement twice during its flight (once on the way up, and once on the way down), and the velocity will clearly be different in each case (it will be positive on the way up, and negative on the way down). So we cannot determine the velocity of this object based on its displacement alone.

CHAPTER 25: DIFFERENTIAL EQUATIONS

A Differential equations
B Solutions of differential equations
C Differential equations of the form $\frac{dy}{dx} = f(x)$
D Separable differential equations
E Slope fields
F Euler's method for numerical integration

Syllabus references: AHL 5.14, ALH 5.15, AHL 5.16

In Section A, students set up a differential equation from a context. This should help them understand the concept of a differential equation.

In Section B, we discuss the solution of differential equations. It is important for students to understand that the solution to the differential equation is an *equation* such as $y = x^2 + x$, rather than a number such as $x = 5$. This understanding will set up students well for the remainder of the chapter. It may be useful to remind students of their integration work in Chapter 22, when they would, for example, find y if $\frac{dy}{dx} = 2x^3 - 4$. When they did this, they were effectively solving a differential equation.

Slope fields are presented in Section E. At the end of the section, students will be asked to construct some simple slope fields of their own. Based on the syllabus, we feel it is unlikely that students will need to construct slope fields in an exam, but we feel the act of constructing slope fields will help students understand them better.

We end the chapter with Euler's method for numerical integration. In the Discussion at the end of Section F, students should find that the accuracy of the solution decreases as the number of iterations increases. This is because the error between the approximation and the exact solution is likely to build as we progress through the iterations.

At the end of the chapter there is an interesting Activity about the spruce budworm. Students who have finished the exercises early should be encouraged to go through it, as it introduces equilibrium points in a 1-dimensional context before we encounter equilibrium points in 2-dimensions when we study phase portraits.

CHAPTER 26: COUPLED DIFFERENTIAL EQUATIONS

A Phase portraits
B Coupled linear differential equations
C Second order differential equations
D Euler's method for coupled equations

Syllabus references: AHL 5.16, AHL 5.17, AHL 5.18

We start this chapter with an exploration of phase portraits. This is really an opportunity for students to "play" with phase portraits, to understand what they represent, and to investigate the different types of equilibrium points that can occur. While the syllabus suggests that we need only consider equilibrium points at the origin, a question in a specimen exam requires students to find other equilibrium points, so we feel that asking students to do this in this section is reasonable.

Students may find it useful to discuss the similarities and differences between phase portraits and the slope fields studied in the previous chapter.

In Section B, we restrict our study to coupled linear differential equations, in which the only equilibrium point occurs at the origin. We go into detail to explain the behaviour of a phase portrait around the equilibrium point, which depends on the eigenvalues of the matrix of coefficients. This seems necessary, as the teacher support material suggests that students may need to sketch a phase portrait from the matrix form. If the eigenvalues are real, students may also need to find a general or particular solution to the system of differential equations.

In Section C, we solve second order differential equations by writing them as coupled linear differential equations. Note that there appears to be an error in the syllabus document which states that "Solutions of $\frac{d^2x}{dt^2} + a\,\frac{dx}{dt} + b = 0$ can also be investigated using the phase portrait method ....". We believe this should be "Solutions of $\frac{d^2x}{dt^2} + a\,\frac{dx}{dt} + bx = 0$ ....".

In Section D we use Euler's method to approximate the solution to coupled differential equations. This includes second order differential equations which are converted to coupled differential equations.

In general, we feel that it is unfortunate that students taking this course will not have enough algebra behind them to truly understand the solutions to these differential equations. However, we hope that students find the treatment of differential equations interesting, and that they enjoy the real-world applications that can be explored. We would particularly encourage students to have a look at the Activity about the SIR model for infectious disease. This is an incredibly rich and engaging Activity, and it would be an excellent basis for a Mathematical Exploration.

CHAPTER 27: DISCRETE RANDOM VARIABLES

- **A** Random variables
- **B** Discrete probability distributions
- **C** Expectation
- **D** Variance and standard deviation
- **E** Properties of $aX + b$
- **F** The binomial distribution
- **G** Using technology to find binomial probabilities
- **H** The mean and standard deviation of a binomial distribution
- **I** The Poisson distribution

Syllabus references: SL 4.7, SL 4.8, AHL 4.14, AHL 4.17

We start the chapter with an introduction to the concept of (discrete) random variables and their probability distributions. If you are going through the Core book and this book in chapter order, it will have been a long time since the students have seen probability. Before starting this chapter, it may be beneficial to briefly revise key probability concepts as they are assumed throughout this chapter, Chapter 28 (The normal distribution), and Chapter 30 (Hypothesis testing).

Section C (Expectation) continues on directly from Section E (Making predictions with probability) from Chapter 11 of the Core book.

In Section C.2, a Discussion asks students whether we would expect a gambling game to be "fair". Students should recognise that we would not expect gambling games to be "fair", otherwise the operator of the game would not make a profit. A useful direction to lead students would be to ask whether the word "fair" in the mathematical sense is equivalent to how the word is used in everyday life, and whether the fact that gambling games are not "fair" implies that the operators are being underhanded or deceptive.

Although variance and standard deviation are not strictly part of the course, we include Section D (Variance and standard deviation) as the concept of variance for a *random variable* is required to understand the variance of a *linear combination of random variables*, which will be covered in Chapter 29.

Since the binomial theorem is not included in the Applications syllabus, we motivate and define the binomial coefficient using Pascal's triangle instead. In calculating binomial probabilities without technology, we encourage students to use Pascal's triangle to help find binomial coefficients.

Finally we end the chapter with Section I (The Poisson distribution). We motivate the Poisson distribution by dividing an interval of time into successively smaller intervals, and considering the limit of a particular binomial random variable. Unfortunately due to the syllabus' omission of the binomial coefficient, we cannot include a proof of the Poisson probability mass function.

Factorial notation is also very briefly introduced here as it appears in the Poisson probability mass function. In worked examples we have used the probability mass function explicitly in probability calculations to demonstrate that the notation should not be particularly difficult to understand. We have also provided calculator instructions to do these calculations with technology if required.

CHAPTER 28: THE NORMAL DISTRIBUTION

- **A** Introduction to the normal distribution
- **B** Calculating probabilities
- **C** The standard normal distribution
- **D** Quantiles

Syllabus references: SL 4.9

In the chapter's introduction, we briefly mention probability density functions as the continuous analogue of probability mass functions. Here, we give the definition that the probability is the area under the curve without further comment or investigation. We are simply using the probability density function as a tool to justify the notion that area under the normal curve = probability later in the chapter, and nothing more.

The first section introduces the normal distribution by focusing on how the normal distribution arises and exploring the *shape* of the distribution. Probability calculations are treated separately in the following section.

Section C (The standard normal distribution), is not in the course. However, we have included it because we feel that it is necessary for understanding some of the concepts in hypothesis testing in Chapter 30.

CHAPTER 29: ESTIMATION AND CONFIDENCE INTERVALS

A Linear combinations of random variables
B The sum of two independent Poisson random variables
C Linear combinations of normal random variables
D The Central Limit Theorem
E Confidence intervals for a population mean with known variance
F Confidence intervals for a population mean with unknown variance

Syllabus references: AHL 4.14, AHL 4.15, AHL 4.16, AHL 4.17

In this chapter we introduce the notion of considering multiple random variables simultaneously. For the first time, we consider statistics such as the sample mean and sample variance as random variables, rather than simple numbers that can be interpreted.

This chapter lays much of the theoretical foundation for the following two hypothesis testing chapters. So it is important to ensure that this chapter is covered well before moving forward.

Much of the material in this chapter has been taken from our previous Statistics Option book. However it has been heavily revised to better reflect:

- the new Applications course
- commonly accepted conventions in mathematical statistics, at a university level and in practice.

The second point is emphasised by the fact that we refer to S_{n-1}^2 as *the* sample variance.

The t-distribution is introduced when we get to Section F (Confidence intervals for the population mean with unknown variance). Despite its non-examinable nature, the t-distribution should not be dismissed as being unimportant or unnecessary as it is vital in understanding how the confidence intervals in this case are constructed, and will reappear later in Chapter 30.

CHAPTER 30: HYPOTHESIS TESTING

A Statistical hypotheses
B The Z-test
C Critical values and critical regions
D Student's t-test
E Paired t-tests
F The two-sample t-test for comparing population means
G Hypothesis tests for the mean of a Poisson population
H Hypothesis tests for a population proportion
I Hypothesis tests for a population correlation coefficient
J Error probabilities and statistical power

Syllabus references: SL 4.11, AHL 4.18

We start the chapter by introducing the student to the terminology used in hypothesis testing:

- Statistical hypotheses: their definition, formulation, and role in a hypothesis test.
- Type I and II error. At this stage we only consider their definitions.

We then introduce the hypothesis testing procedure via the Z-test. The Z-test is not *strictly* in the course. However we choose to include it because it is necessary for when we consider critical values and power calculations for tests involving the normal distribution (namely the Z-test) later in the chapter. Also, we believe that hypothesis testing is best introduced using the normal distribution, which should be very familiar to students by this stage.

At this point, it is important that students clearly understand what each component and step of the testing procedure means, as these concepts will reappear in every following section.

Section C (Critical values and critical regions) introduces the concept of critical regions and values. Here we focus on the normal distribution/Z-test exclusively, but critical value type questions will reappear throughout the t-test sections.

The main purpose of the discussion at the end of Section C is to recognise that while the procedures are equivalent, in general the acceptance region is not the same as the confidence interval with the corresponding confidence level. This is because the former is centred at μ_0 and the latter is centred at $\bar{x}$. It is important to highlight the similarities of confidence intervals with critical regions and acceptance regions, but be careful not to confuse the two concepts. This is the main reason why confidence intervals is created separately from hypothesis testing.

Sections D and E explore the t-test for the one-sample and two-sample cases respectively.

Sections F through G deal with hypothesis tests for other population parameters and distributions. The concepts in these sections are exactly the same as for the Z-test and the t-tests, just in different contexts.

Although we provide procedures for each test type, it should be emphasised that the student should not be remembering them as completely separate procedures. Being able to identify the common elements, and how and *why* the procedures differ, are paramount to successfully understand the concepts in this chapter.

Students should be encouraged to view the online flowchart at the end of the chapter, which will help students determine which hypothesis test should be used, and which variable they will need. If a student chooses these correctly, then their calculator should do the rest.

CHAPTER 31: χ^2 HYPOTHESIS TESTS

A The χ^2 goodness of fit test
B Estimating distribution parameters in a goodness of fit test
C Critical regions and critical values
D The χ^2 test for independence

Syllabus references: SL 4.11, AHL 4.12

In this chapter, we introduce the notion of using hypothesis tests to test claims about how well a *distribution* fits or *models* data or a population. This contrasts with the testing of parameters of predetermined distributions, which was the focus in Chapter 30. However, most of this chapter is still written in the same style and structure as Chapter 30, emphasising the fact that the general procedure is still the same, only the context is different.

We start exploring this idea with the χ^2 goodness of fit (GOF) test, instead of the χ^2 test of independence. We chose to do the GOF test first because it is the most general and the latter is just a special case of the former. In this section we introduce the concepts of observed and expected frequencies, and degrees of freedom (df).

At this point only the p-value is considered for the decision rule because Section B (Estimating distribution parameters in a goodness of fit test) follows on directly from Section A. Here, we talk about df more in depth, and consider the named distributions (binomial, Poisson, and normal) where some of the parameters are unknown. It should be noted that when the GOF test is applied to normal distributions, the lowest and greatest class intervals need to be extended infinitely in order for the probabilities to sum to 1. This is highlighted in Example 3.

We only briefly mention the χ^2-distribution for conceptual understanding of the p-value and critical values. For all calculations, we use the built-in calculator functions to calculate p-values and a printable table of values to determine critical values.

Finally we cover the χ^2 test for independence in Section D. As noted previously, this test is just a special case of the GOF test. The distribution and hence critical values are exactly the same as those used in the previous section. Only how the df (degrees of freedom) is calculated appears to have changed. The relationship between the df for GOF and the test for independence is highlighted in the Discussion at the end of the section:

$$\begin{aligned}\text{df} &= (r-1)(c-1)\\ &= rc - r - c + 1\\ &= rc - (r+c-2) - 1\\ &= \text{number of categories} - \text{number of estimated parameters} - 1\end{aligned}$$

- The total number of "categories" is the number of probabilities $=$ number of rows $\times$ number of columns.
- The "-1" comes from the fact that probabilities have to sum to 1.
- Each row and column sum is an estimated parameter. 2 is subtracted because the total of the row sums must equal the overall total. Likewise for the column sums.