Mathematics: Applications and Interpretation SL

Chapter summaries

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CHAPTER 1: APPROXIMATIONS AND ERROR

A Rounding numbers
B Approximations
C Errors in measurement
D Absolute and percentage error

Syllabus references: SL 1.6

The first two sections of this chapter are likely to be quite straightforward for most students. Indeed, much of the material from the Mathematics: Core Topics SL book will have assumed a familiarity with rounding numbers. Teachers should feel free to skip these early sections if the students are already comfortable with this material.

In Section B, a Discussion asks students about the accuracy of one figure approximations. Students should find that the example given is accurate because one number is rounded up and the other is rounded down, and because both numbers are quite close to their one figure approximation. Other approximations will not be as accurate, for example in the approximation of 14×24, both numbers are rounded down when approximated by one figure, and both are relatively far from their one figure approximation.

In Example 10, we find the maximum percentage error that results from rounding. To do this, we first find the boundary values for the measurement given the rounded value. We then find the percentage error associated with each of the boundary values, and the largest value gives us the maximum percentage error. In practice, the maximum percentage error will always arise when the actual value is the lower boundary value, but we feel that this is not obviously the case, so it is more intuitive to students to try both boundary values.

CHAPTER 2: LOANS AND ANNUITIES

A Loans
B Annuities

Syllabus references: SL 1.7

This chapter follows on from the work done in Chapter 5 of the Mathematics: Core Topics SL book, where technology was used to solve problems involving compound interest investments. In this chapter we consider the more complex cases of loans and annuities, where regular payments or withdrawals are made from the account. It would be useful for students to see loans and annuities as the "reverse models" of each other, in which the bank and the individual are essentially swapping roles.

When using the TVM solver on the calculator, students should be made aware at this point that N is the total number of payments.

In Chapter 5 of the Core Topics SL book, we define N as "the number of compounding periods". This is the only sensible definition of N at this point, because we are dealing with investments with no regular payments or repayments. Students are instructed to give P/Y and C/Y the same value, which is the number of compounding periods per year. This allows the calculator to use N and P/Y to work out the number of years, even though there are no payments.

However, now that we are dealing with problems involving payments and repayments, students should be made aware that N is the number of payments, not the number of compounding periods. For the vast majority of problems, P/Y and C/Y will be equal, so there will not be an issue. However, in cases where P/Y and C/Y are different (for example, a person making quarterly contributions into an account which pays interest compounded monthly), students should make sure that N refers to the total number of payments.

If students are obtaining answers that differ from those in the back of the book by only a few cents, it may be due to rounding of intermediate values in calculations. For example, loan repayments are always rounded up to the next cent, rather than to the nearest cent, because if the repayment was rounded down, the loan would not be completely repaid in the specified time. The rounding used in worked examples should provide a good guide for students.

Section B contains a Discussion about superannuation, and why a government might make superannuation compulsory. In forming their answers, students should consider the consequences of a person reaching retirement age with no money saved, not only for that individual, but also for the broader society.

CHAPTER 3: FUNCTIONS

- A Relations and functions
- B Function notation
- C Domain and range
- D Graphs of functions
- E Sign diagrams
- F Transformations of graphs
- G Inverse functions

Syllabus references: SL 2.2, SL 2.3, SL 2.4

Although much of the content of this chapter is shared with the Analysis and Approaches SL course, this chapter has been put in the Mathematics: Applications and Interpretation SL book because this course requires a more informal treatment of functions. In particular, inverse functions are explained geometrically (as a reflection in the line $y = x$) rather than algebraically. There is also a more informal treatment of asymptotes, although students are expected to identify asymptotes from a graph.

At the end of Section A, there is a Discussion about the Opening Problem. Students should discover that the car park charges is indeed a function, as for each unit of time there is exactly one charge. However, for a given charge, there are infinitely many possible times. It would be useful to refer back to this when inverse functions are introduced. Since the car park function is not one-to-one, the function is not invertible. This should feel consistent with the findings of the Discussion, since we cannot specify a function which determines the time someone has parked for, given their charge.

Although transformations of graphs are not explicitly included in the syllabus, in this chapter we will consider vertical translations, and vertical and horizontal stretches of functions, as these will help students in the study of quadratic, exponential, and trigonometric functions in later chapters. Reflections of functions are also presented in preparation for reflecting functions in the line $y = x$ to give inverse functions.

Sign diagrams are presented in this chapter, as they will be useful in calculus.

CHAPTER 4: MODELLING

- A The modelling cycle
- B Linear models
- C Piecewise linear models
- D Systems of equations

Syllabus references: SL 1.8, SL 2.5, SL 2.6

In this chapter, students are introduced to the concept of a mathematical model. We see that we can take real-life problems, and then make assumptions about the situation in order to simplify it to a form which can be represented mathematically. However, it is important to assess how reasonable the assumptions are, and to understand that the assumptions affect the accuracy of the final answer. We must therefore strike a balance between simplicity and accuracy.

In Section B, we use linear models to distinguish between exact and approximate models. If the data appears to follow a linear trend, we may use a linear model to approximate the situation. The work done in this section is a precursor to what is done in Chapter 5, where we consider approximate linear models more formally.

In the final section, we use technology to find unknown coefficients in a model. This is done by using given information to create a 2×2 or 3×3 system of simultaneous equations. This method will be used in future chapters to determine quadratic, exponential, and trigonometric models.

CHAPTER 5: BIVARIATE STATISTICS

A Association between numerical variables
B Pearson's product-moment correlation coefficient
C Line of best fit by eye
D The least squares regression line
E Spearman's rank correlation coefficient

Syllabus references: SL 4.4, SL 4.10

Most of this chapter is similar in style and presentation as in previous books and the MYP series.

In Section D, there is a Discussion about finding the line which minimises the horizontal, rather than vertical, distances between the data points and the line. Students should conclude that this line is different from the least squares regression line, and is found by swapping the variables used in the regression. In other words, we find the regression line of x against y, rather than y against x. This line would be more reliable for estimating y given a value of x.

Section E (Spearman's rank correlation coefficient) is new. We motivate the need for the rank correlation by focussing on the *direction* of a trend that may not necessarily be linear.

We would expect classes to reach the end of this chapter by the end of the first year.

CHAPTER 6: QUADRATIC FUNCTIONS

A Quadratic functions
B Graphs from tables of values
C Axes intercepts
D Graphs of the form $y = ax^2$
E Graphs of quadratic functions
F Axis of symmetry
G Vertex
H Finding a quadratic from its graph
I Intersection of graphs
J Quadratic models

Syllabus references: SL 2.4, SL 2.5

Since solving quadratic equations is not part of the Applications and Interpretation SL syllabus, this chapter has much more of a technology focus than the equivalent chapter in the Mathematics: Analysis and Approaches SL book. Technology is used to find the x-intercepts of a quadratic function, and more generally to find the value(s) of x for a given value of y.

When finding a quadratic given information about the graph, students will sometimes need to solve a system of linear equations, which was studied in the Modelling chapter. This is done in non-contextual situations in Section H, and then in context in Section J. Although some quadratic models were considered in the Modelling chapter, here we extend that work now that students have the tools to consider properties of the quadratic model such as the axis of symmetry and vertex.

In the Research task in Section J, students must find the correct model for a free-hanging rope. They should find that the correct model is a catenary. This model is quite complex, so this is a good opportunity to connect back with the Modelling chapter. Students can weigh up the advantages and disadvantages of using the simpler quadratic model, or the more correct catenary model, when describing a free-hanging rope.

CHAPTER 7: DIRECT AND INVERSE VARIATION

A Direct variation
B Powers in direct variation
C Inverse variation
D Powers in inverse variation
E Determining the variation model
F Using technology to find variation models

Syllabus references: SL 2.5

The material in Sections A to D is unlikely to be assessed directly, but is important so that students understand the concept of direct and inverse proportion. That being said, this material can be progressed through quickly if the students are familiar with it.

In the Dicussion at the end of Section D, students should conclude that z is proportional to x^2.

In Section E, students are generally given the type of variation relationship that exists between two variables (for example, y is proportional to x^3), and must use given data to find the proportionality constant. The models in this section are exact.

In Section F, students must use technology to find the best variation model connecting two variables, given a set of data. Students should be familiar with this process from their work on linear regression in Chapter 5. As is the case in Chapter 5, students should use the correlation coefficient r to determine how well the model fits the data. Students should also think about the context the data is given in, and use the context to sensibly round the parameters of the model. For example, we may suspect a shape's *area* will be proportional to the square of its *perimeter* as it is enlarged. If the calculator returns a model with power 2.0001, this power should simply be rounded to 2.

In the Discussion in Section F, we should expect that the mass of the ball bearings is proportional to radius^3 because, assuming uniform density of the ball bearings, we would expect their mass to be proportional to their volume, which in turn is proportional to radius^3.

CHAPTER 8: EXPONENTIALS AND LOGARITHMS

A Exponential functions
B Graphing exponential functions from a table of values
C Graphs of exponential functions
D Exponential equations
E Growth and decay
F The natural exponential
G Logarithms in base 10
H Natural logarithms

Syllabus references: SL 1.5, SL 2.5

In this chapter students are introduced to exponential functions. After briefly encountering asymptotes in Chapter 3, students will investigate them in more detail here.

Since solving exponential equations algebraically is not part of the Applications and Interpretation SL course, students will use graphical methods to find the value of x for a given value of y.

Students will once more use technology to find unknown coefficients of exponential functions, by using known information to construct systems of equations.

The work done in transforming functions in Chapter 3 will be useful here, as students should be able to recognise that $y = 2^{-x}$ is a reflection of $y = 2^x$ in the y-axis.

In the Discussion at the end of Section C, students are asked why we specify a positive base number $a \neq 1$. As a starting point, they should recognise that if $a = 1$, the graph simply becomes the horizontal line $y = 1$, which is not an exponential function. They should also find that, for negative values of a, the graph does not form a smooth curve, but instead bounces between positive and negative values. Students may find the graphing package useful for exploring functions like $y = (-2)^x$. They should realise that essentially two subgraphs are formed: the graph of $y = 2^x$, and $y = -(2^x)$. From their work on negative bases in the Mathematics: Core Topics SL book, they should understand that $y = (-2)^x$ will be positive for even integer values of x, and negative for odd integer values of x. By experimenting with fractional values of x, they may find $(-2)^x$ is defined when x is rational with an odd denominator (whether it is positive or negative depends on whether the numerator is even or odd), and undefined when x is rational with even denominator. However, without a good grounding in rational exponents, it is unlikely they will have the mathematics to explain why this is occurring.

In Section F, students are introduced to the natural exponential e. Since calculus involving e is not included in this course, it is difficult to motivate the use and importance of e We present students with an Investigation to see how e appears in continuously compounding interest. Transformation of graphs again comes into play here, as students should recognise that $y = e^{rx}$ is a horizontal stretch of $y = e^x$ with scale factor $\frac{1}{r}$.

The chapter ends with a brief introduction to logarithms. It is perhaps unfortunate that laws of logarithms are not included in this syllabus, as there is not much that students can do with the logarithms. The final question of the chapter guides students to the idea that $y = \ln x$ is the inverse function of $y = e^x$.

CHAPTER 9: TRIGONOMETRIC FUNCTIONS

A The unit circle
B Periodic behaviour
C The sine and cosine functions
D General sine and cosine functions
E Modelling periodic behaviour

Syllabus references: SL 2.5

This chapter begins by using the unit circle to extend the definition of trigonometric ratios to *all* angles.

We then use the transformations studied in Chapter 3 to construct graphs of trigonometric functions. Radians are not part of this course, so all trigonometric functions will be given in terms of angles in degrees. We use the modulus symbol to talk about the amplitude of a trigonometric function. If needed, some material on modulus is provided in the Background Knowledge.

In Section D, a Discussion asks students how the graphs of sine and cosine are related. Students should find that $y = \sin x$ is a translation of $y = \cos x$ horizontally through $90°$. However, the syllabus states that students are not expected to translate between sine and cosine in their study of these functions.

When using information to construct a trigonometric model, students should be careful to consider all the information given. For example, when modelling the height of a rotating windmill blade, students will need to consider the starting position of the blade, and whether the blade is rotating clockwise or anticlockwise.

CHAPTER 10: DIFFERENTIATION

A Rates of change
B Instantaneous rates of change
C Limits
D The gradient of a tangent
E The derivative function
F Differentiation
G Rules for differentiation

Syllabus references: SL 5.1, SL 5.3

This chapter provides students with their first look at differential calculus. Although much of the calculus content is common between the SL courses, we expect the classes will be separated by the time they encounter calculus. Having the calculus chapters in the separate books allows a more targeted approach to calculus for each course.

This chapter begins with rates of change, which is used to motivate an informal study of limits. A blended learning investigation, which combines limits, previous results, and technology, is used to explore the instantaneous rate of change for a curve.

The Discussion at the end of Section C invites students to ponder the existence of limits. To answer this, it may help students to return to how we define the limit of a function. Is there any guarantee that $f(x)$ will get closer and closer to a particular value as x gets closer and closer to a? When considering the graph of $f(x) = \frac{1}{x}$, students should find that the graph approaches $-\infty$ as x approaches 0 from the left, and the graph approaches ∞ as x approaches 0 from the right. Since the graph does not approach a particular value as x approaches 0, $f(x) = \frac{1}{x}$ does not have a limit as x approaches 0.

Although differentiation from first principles is not explicitly in the syllabus, we feel that it is essential for understanding the process of differentiating a function, and there is no point in studying limits if we do not consider first principles.

At the end of Section F, there is a Discussion which asks whether a function always has a derivative function, and whether the domains of a function and its derivative are always the same. To help guide students, they should be encouraged to consider the definition of "function" as broadly as possible. For example, a collection of disconnected points is considered to be a function, as long as no pair of points share the same x-coordinate. A function does not need an "equation" which defines its set of points. Students should also consider functions such as $f(x) = \sqrt{x}$, and discuss whether this function has a limit as x approaches 0, given that the function is undefined for $x < 0$.

Since rules such as the product and quotient rules are not part of this course, we have included a section entitled "Rules of differentiation" at the end of this chapter, rather than it being a chapter of its own as in the Mathematics: Analysis and Approaches SL book. This section covers the derivative of x^n, as well as the linearity of differentiation. These rules are sufficient to differentiate polynomial functions.

CHAPTER 11: PROPERTIES OF CURVES

A Tangents
B Normals
C Increasing and decreasing
D Stationary points

Syllabus references: SL 5.2, SL 5.4, SL 5.6

This chapter allows students to apply the calculus they have learnt to discover the properties of curves.

This chapter is considerably shorter than the corresponding chapter in the Mathematics: Analysis and Approaches SL book. This is because, in this course, students can only differentiate polynomial functions, and inflection points are not part of this course. Only stationary inflections are mentioned, in the context of stationary points.

Students should be encouraged to think of the concepts of increasing and decreasing in terms of intervals, rather than at a particular point. This will help students understand why, for example, the graph of $y = x^2$ is increasing for $x \geqslant 0$, and decreasing for $x \leqslant 0$.

In Section D we return to sign diagrams, and see how the sign diagram of $f'(x)$ can be used to determine the nature of the stationary points of $f(x)$.

As outlined in the syllabus, instructions are given to use technology to sketch the graph of $f'(x)$, which can then be used to solve $f'(x) = 0$. However, students should be encouraged to use an algebraic approach to find stationary points where possible, especially since we can use technology to find local maxima and minima directly from the graph of $f(x)$.

CHAPTER 12: APPLICATIONS OF DIFFERENTIATION

A Rates of change
B Optimisation
C Modelling with calculus

Syllabus references: SL 2.5, SL 5.1, SL 5.7

In this final chapter of differential calculus we explore some of its real world applications. The important skill in this chapter is to take the calculus techniques learnt in previous chapters, apply them to real life problems, and interpret the results in the context of that problem.

Students should be encouraged to keep in mind the constraints imposed by the context of the problem, and to make sure their solution makes sense in this context.

The Discussion at the end of Section A asks students to consider the volume V and radius r of a sphere, and to think about why $\frac{dV}{dr}$ is proportional to its surface area A. A useful way for students to approach this problem is to think about what happens to the volume of a sphere when the radius increases by a very small amount. The amount by which the volume has increased is equal to the volume of the "shell" formed by removing the "old" sphere from the centre of the "new" sphere.

This shell approximates the surface area of the sphere. Although $\frac{dV}{dr} = A$ in this case, students should see with the case of the cube with side length x that $\frac{dV}{dx} \neq A$, but $\frac{dV}{dx}$ is proportional to A.

In Section C, we use known information to find unknown coefficients in models, like we have done in previous chapters. Now, however, some of the information is given in the context of the derivative, rather than of the model itself. In particular, this Section gives students an opportunity to model with cubic functions.

CHAPTER 13: INTEGRATION

A Approximating the area under a curve
B The Riemann integral
C The Fundamental Theorem of Calculus
D Antidifferentiation and indefinite integrals
E Rules for integration
F Particular values
G Definite integrals
H The area under a curve

Syllabus references: SL 5.5, SL 5.8

This material may be challenging for some students, as integral calculus was not in the previous Mathematical Studies SL course. However, we believe the focus here will be on positive functions and the area under a curve, and for this application we believe integration is actually more intuitive to weaker students than differentiation.

We begin with some numerical methods for finding the area under a curve, which is consistent with the history of integration, and so ties in well with IB thinking.

Section B is entitled "The Riemann Integral", which may sound complicated, but all we are really doing is giving some integral notation to describe the area under a curve. In this section we explore what happens as we consider lower and upper rectangles with infinitely many subintervals, using our work with limits at infinity.

Only then do we discuss the Fundamental Theorem of Calculus, which links the area under a curve with the differentiation we have studied previously. The Fundamental Theorem of Calculus therefore gives us motivation to antidifferentiate. We generalise from examples to give basic rules for integration, then tie it all together by returning to definite integrals and the area under a curve.

Once we have established that integration is the reverse process of differentiation, we use the rules for differentiation in reverse to develop the rules for integration. In Section E, students are asked to discuss why we specify that our rule for integrating x^n is not valid for $n = -1$. Students should conclude that substituting $n = -1$ into this rule would result in a division by zero.

In some instances, extra information about the original function is included, allowing us to determine the constant of integration. Occasionally simultaneous equations will be required to find multiple unknowns.

Once we have established the rules for integration, we now have more tools to calculate definite integrals, and to explore the relationship between definite integrals and areas. The syllabus specifies that technology must be used for some definite integrals. To this end, we have included calculator instructions, screenshots, and exercises which allow students to practise finding definite integrals using technology.

CHAPTER 14: DISCRETE RANDOM VARIABLES

A Random variables
B Discrete probability distributions
C Expectation
D The binomial distribution
E Using technology to find binomial probabilities
F The mean and standard deviation of a binomial distribution

Syllabus references: SL 4.7, SL 4.8

We start the chapter with an introduction to the concept of (discrete) random variables and their probability distributions. If you are going through the Mathematics: Core Topics SL book and this book in chapter order, it will have been a long time since the students have seen probability. Before starting this chapter, it may be beneficial to briefly revise key probability concepts as they are assumed throughout this chapter, Chapter 15 (The normal distribution), and Chapter 16 (Hypothesis testing).

Section C (Expectation) continues on directly from Section J (Making predictions with probability) from Chapter 10 of the Mathematics: Core Topics SL book.

In Section C.2, a Discussion asks students whether we would expect a gambling game to be "fair". Students should recognise that we would not expect gambling games to be "fair", otherwise the operator of the game would not make a profit. A useful direction to lead students would be to ask whether the word "fair" in the mathematical sense is equivalent to how the word is used in everyday life, and whether the fact that gambling games are not "fair" implies that the operators are being underhanded or deceptive.

Since the binomial theorem is not included in the Applications and Interpretation SL syllabus, we motivate and define the binomial coefficient using Pascal's triangle instead. In calculating binomial probabilities without technology, we encourage students to use Pascal's triangle to help find binomial coefficients.

CHAPTER 15: THE NORMAL DISTRIBUTION

A Introduction to the normal distribution
B Calculating probabilities
C Quantiles

Syllabus references: SL 4.9

In the chapter's introduction, we briefly mention probability density functions as the continuous analogue of probability mass functions. Here, we give the definition that the probability is the area under the curve without further comment or investigation. We are simply using the probability density function as a tool to justify the notion that area under the normal curve = probability later in the chapter, and nothing more.

The first section introduces the normal distribution by focusing on how it arises and exploring its *shape*. Probability calculations are treated separately in the following section.

CHAPTER 16: HYPOTHESIS TESTING

A Statistical hypotheses
B Student's t-test
C The two-sample t-test for comparing population means
D The χ^2 goodness of fit test
E The χ^2 test for independence

Syllabus references: SL 4.11

We start the chapter by introducing the student to the terminology used in hypothesis testing. In particular, we first focus on statistical hypotheses: their definition, formulation, and role in a hypothesis test.

We then introduce the hypothesis testing procedure via the t-test. At this point, it is important that students clearly understand what each component and step of the testing procedure means, as these concepts will reappear in each following section.

Although the t-distribution is not in the Applications and Interpretation SL course, we have included a brief definition of it for the sake of conceptual understanding, as the concept of a test statistic's *distribution* is fundamental to one's understanding of the p-value as a probability. We are aware that students and teachers may skip the relevant Investigation and theory due to its non-examinable nature. Thus, we have provided an example and calculator instructions for conducting the t-test using technology.

Section C (Comparing population means) looks at the two-sample version of the t-test. The calculations in this section are done completely with technology, unlike in Section B. We felt that with the two different cases for pooling data, giving the formulae for the test statistics of each case was counter-productive to conceptual understanding.

In Section D (The χ^2 goodness of fit test), we introduce the χ^2 goodness of fit test and the concepts of observed and expected frequencies. Both p-values and critical values are covered for the decision rule. The Activity at the end of the Section shows how the χ^2 goodness of fit test can be used in the context of *modelling* - the primary theme of this course. Although subtle, it serves as a reminder that probability, statistics, and inference are just other facets of modelling.

Section E (The χ^2 test for independence) should be familiar to teachers who have previously taught the Mathematical Studies SL course. However, this section has seen a major overhaul from its incarnation in the Mathematical Studies SL book. It has been rewritten to be more like the preceding sections and follows on directly from Section D. Since the χ^2 test for independence is just a special case of the χ^2 goodness of fit test, it should be noted that the distribution and hence critical values are exactly the same as those used in the previous section. Only how the df (degrees of freedom) is calculated has changed.

Although we provide procedures for each test type, it should be emphasised that the student should not be remembering them as completely separate procedures. Being able to identify the common elements, and how and *why* the procedures differ are paramount in successfully understanding the concepts in this chapter.

It should also be noted that not all calculators will have the relevant hypothesis test functions. For example, the HP Prime Graphing calculator lacks functions to perform the χ^2 goodness of fit test and the χ^2 test for independence.

CHAPTER 17: VORONOI DIAGRAMS

- A Voronoi diagrams
- B Constructing Voronoi diagrams
- C Adding a site to a Voronoi diagram
- D Nearest neighbour interpolation
- E The Largest Empty Circle problem

Syllabus references: SL 3.6

We conclude this book with a study of Voronoi diagrams, a topic which is likely to be unfamiliar to many teachers and students.

We begin with presenting students with Voronoi diagrams, and asking students to interpret the diagrams. We hope this will allow students to get used to the concept and the terminology, and help develop an intuition as to how the diagrams should look, before the students are asked to construct a Voronoi diagram of their own.

We then guide students through the construction of Voronoi diagrams, including adding an extra site to a diagram, finding a missing site or edge, and finding the largest empty circle within a diagram.

At the end of Section D, students are asked to discuss the advantages and disadvantages of nearest neighbour interpolation. In their discussion, students should consider its ease of use, and its accuracy. Students should also note that the context in which it is used is an important consideration. In some situations, it is more important to get a reasonable estimate quickly, and in other situations, accuracy is more important. When discussing more accurate ways to perform the interpolation, students should note that the nearest neighbour algorithm does not take into account the relative distance of a point from the other sites other than its nearest site. For example, if a point is closest to site B, but is almost as close to site A, the algorithm should take this into account, and should consider the value at site A as well as at site B.

We hope that the contextual nature of these problems will make this chapter engaging for students.