IB Mathematics HL

SMARTPREP® | Haese Mathematics

Topic 1 | Sequences and series

Theory card T Question cards

Front side

A **number sequence** is a set of numbers defined by a rule. The rule is often a formula for the **general term** or ***n*th term** of the sequence.

A sequence which continues forever is called an **infinite sequence**.

A sequence which terminates is called a **finite sequence**.

Arithmetic Sequences

- each term differs from the previous one by the same fixed number
- $u_{n+1} - u_n = d$ for all n, where d is the **common difference**
- the general nth term is $u_n = u_1 + (n-1)\,d$.

Geometric Sequences

- each term is obtained from the previous one by multiplying by the same non-zero constant
- $\dfrac{u_{n+1}}{u_n} = r$ for all n, where r is the **common ratio**
- the general nth term is $u_n = u_1 r^{n-1}$.

Turn around

Back side

Compound interest at $i\%$ per compounding period gives a geometric sequence with common ratio $\left(1+\frac{i}{100}\right)$, which is raised to the power n, where n is the number of compounding periods.

Series

A **series** is the addition of the terms of a sequence.

For a **finite series** with n terms, the sum $S_n = \sum_{k=1}^{n} u_k = u_1 + u_2 + \dots + u_n$.

For a **finite arithmetic series**, $S_n = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(2u_1 + (n-1)d)$.

For a **finite geometric series**, $S_n = \dfrac{u_1(r^n - 1)}{r-1}$, $r \neq 1$.

For an **infinite series**, the sum $\sum_{k=1}^{\infty} u_k$ can only be calculated in some cases.

The sum of an **infinite geometric series** is $S = \dfrac{u_1}{1-r}$ provided $|r| < 1$.

If $|r| > 1$ the series is **divergent**.

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Question side

a An arithmetic series has terms $u_5 = 11$ and $u_{12} = 53$.

- **i** Find the first term u_1 and common difference d.
- **ii** Find the 22nd term u_{22}.
- **iii** Find the sum of the first 22 terms of the series.

b An infinite geometric series is defined by $\sum_{k=1}^{\infty} 2\left(\frac{1}{4}\right)^k$.

- **i** Find the first term u_1 and common ratio r.
- **ii** Find the sum of the series.

Turn around

Answer side

a **i** $u_5 = 11$ and $u_{12} = 53$

$\therefore\ u_1 + 4d = 11$ and $u_1 + 11d = 53$

$\therefore\ (u_1 + 11d) - (u_1 + 4d) = 53 - 11$

$\therefore\ 7d = 42$

$\therefore\ d = 6$

and $u_1 = 11 - 4d = -13$

ii $u_n = u_1 + (n-1)d$

$\therefore\ u_{22} = u_1 + 21d = -13 + 21(6) = 113$

iii $S_n = \frac{n}{2}(u_1 + u_n)$

$\therefore\ S_{22} = \frac{22}{2}(-13 + 113) = \frac{22}{2} \times 100 = 1100$

b **i** $\sum_{k=1}^{\infty} 2\left(\frac{1}{4}\right)^k$ has general term $u_n = 2\left(\frac{1}{4}\right)^n$

$= \frac{1}{2}\left(\frac{1}{4}\right)^{n-1}$

$\therefore\ u_1 = \frac{1}{2}$ and $r = \frac{1}{4}$.

ii Since $|r| < 1$, the series converges.

$S = \frac{u_1}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$

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Question side

a Tobias invested €1600 on December 1st 2015. His investment earns fixed interest of 4.8% per annum, compounded annually.

i What will the investment be worth on December 1st 2022?

ii How many years will it take for the investment to reach €5000?

b Consider the series $\sum_{k=1}^{\infty} 8(x+1)^{k-1}$.

i For what values of x will the series converge?

ii Evaluate the sum of the series when $x = -0.2$.

Turn around

Answer side

a Tobias' account balance forms a geometric sequence with $u_1 = 1600$ and $r = \left(1 + \frac{4.8}{100}\right) = 1.048$.

i On December 1st 2022, 7 compounding periods have passed.

$\therefore\ u_8 = u_1 \times r^7$

$= 1600 \times 1.048^7 \approx €2221.51$

ii The investment reaches €5000 when

$1600 \times 1.048^n = 5000$

$\therefore\ 1.048^n = \frac{5000}{1600} = \frac{25}{8}$

$\therefore\ n \approx 24.3$ {using technology}

$\therefore$ it will take 25 years for the investment to reach €5000.

b **i** The series is geometric with first term $u_1 = 8$, and common ratio $r = x + 1$.

The series converges provided $|r| < 1$

$\therefore\ -2 < x < 0$

ii When $x = -0.2$, the series converges with

sum $S = \frac{u_1}{1-r} = \frac{8}{1-(-0.2+1)} = \frac{8}{0.2} = 40$

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Question side

Find the sum of all integers between 30 and 90 (inclusive) which are **not** divisible by 3.

Turn around

Answer side

The sum of the integers between 30 and 90 which are *not* a multiple of 3, is $(30 + 31 + 32 + ... + 89 + 90)$
$-(30 + 33 + ... + 87 + 90)$.

Now, $30 + 31 + 32 + ... + 89 + 90$ is an arithmetic series with $u_1 = 30$, $n = 61$, and $u_n = 90$.

$$S_n = \frac{n}{2}(u_1 + u_n)$$

$$\therefore \quad S_{61} = \frac{61}{2}(30 + 90) = 3660$$

Now, $30 + 33 + ... + 87 + 90$ in an arithmetic series with $u_1 = 30$, $n = 21$, and $u_n = 90$

$$\therefore \quad S_{21} = \frac{21}{2}(30 + 90) = 1260$$

$$\therefore \quad \text{the required sum} = 3660 - 1260 = 2400$$

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Question side

5 |? **Topic 1 | Sequences and series**

A sequence is defined by $u_n = 8\left(\frac{3}{4}\right)^{n-1}$.

a Prove that the sequence is geometric.

b Find the 4th term in rational form.

c Find, correct to 3 decimal places where appropriate:

i $\sum_{n=1}^{\infty} u_n$ **ii** $\sum_{n=1}^{15} u_n$

Turn around

Answer side

5 |! **Topic 1 | Sequences and series**

a $\frac{u_{n+1}}{u_n} = \frac{8\left(\frac{3}{4}\right)^n}{8\left(\frac{3}{4}\right)^{n-1}}$

$= \frac{3}{4}$ for all n

$\therefore$ the sequence is geometric with common ratio $r = \frac{3}{4}$.

b $u_4 = 8\left(\frac{3}{4}\right)^3$

$= 8\left(\frac{27}{64}\right)$

$= \frac{27}{8}$

c **i** $\sum_{n=1}^{\infty} u_n = \frac{u_1}{1-r}$

$= \frac{8\left(\frac{3}{4}\right)^0}{1-\frac{3}{4}}$

$= \frac{8}{\frac{1}{4}}$

$= 32$

ii $\sum_{n=1}^{15} u_n = S_{15}$

Now, $S_n = \frac{u_1(r^n - 1)}{r-1}$

$\therefore S_{15} = \frac{8\left(\left(\frac{3}{4}\right)^{15} - 1\right)}{\frac{3}{4} - 1}$

≈ 31.572

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Question side

6 | ? **Topic 1 | Sequences and series**

a A geometric sequence has consecutive terms $k-2$, $2k$, and k^2. Find the value of k.

b A sequence is defined by $u_n = \dfrac{3-2n}{4}$, $n \in \mathbb{Z}^+$.

i Prove that the sequence is arithmetic.

ii Find the 20th term.

iii Find $\displaystyle\sum_{n=20}^{30} u_n$

c Aretha deposits €120 in an account on February 1. The account pays compound interest of 3.4% per annum, compounding monthly on the last day of each month.

i Find the value of the fund after interest is deposited on May 31.

ii Write an expression for the value of the fund after k years.

iii Hence find the value of the fund after 15 years.

Unfold

a Since they are consecutive terms, $\dfrac{k^2}{2k} = \dfrac{2k}{k-2}$ {equating ratios}

$\therefore\ k^2(k-2) = (2k)^2$

$\therefore\ k^3 - 2k^2 = 4k^2$

$\therefore\ k^3 - 6k^2 = 0$

$\therefore\ k^2(k-6) = 0$

If $k=0$, the terms of the sequence are $-2, 0, 0$, which are not geometric.
If $k=6$, the terms of the sequence are $4, 12, 36$, which are geometric with common ratio 3 $\quad\therefore\ k=6$

b **i** $u_n = \dfrac{3-2n}{4}$

$\therefore\ u_{n+1} - u_n = \dfrac{3-2(n+1)}{4} - \dfrac{3-2n}{4}$

$= \dfrac{3-2n-2-3+2n}{4}$

$= -\frac{1}{2}$ for all $n \in \mathbb{Z}^+$

$\therefore$ consecutive terms differ by $-\frac{1}{2}$, so the sequence is arithmetic with $d = -\frac{1}{2}$.

ii $u_1 = \dfrac{3-2(1)}{4} = \frac{1}{4}$ and $d = -\frac{1}{2}$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_{20} = \frac{1}{4} + 19\left(-\frac{1}{2}\right) = -\frac{37}{4}$

iii $\displaystyle\sum_{n=20}^{30} u_n = S_{30} - S_{19}$

$= \frac{30}{2}\left[2\left(\frac{1}{4}\right) + 29\left(-\frac{1}{2}\right)\right] - \frac{19}{2}\left[2\left(\frac{1}{4}\right) + 18\left(-\frac{1}{2}\right)\right]$

$= (-210) - (-80.75)$

$= -129.25$

c Rate per month $= \frac{3.4}{12}\% = \frac{17}{60}$. Let $R = \frac{17}{60} \div 100 = \frac{17}{6000}$

i After 4 months, her amount is
$120(1+R) + 120(1+R)^2 + 120(1+R)^3 + 120(1+R)^4$
which is geometric with $u_1 = 120(1+R)$ and $r = 1+R$.

So, $S_4 = 120(1+R)\left[\dfrac{(1+R)^4 - 1}{1+R-1}\right] \approx €483.41$

ii After k years or $12k$ months,

$S_{12k} = 120(1+R)\left[\dfrac{(1+R)^{12k} - 1}{R}\right]$

$\therefore\ S_{12k} \approx 42\,472.9\left([1.002\,833]^{12k} - 1\right)$

iii After 15 years, $k = 15$

$S_{180} \approx 42\,472.9\left([1.002\,833]^{180} - 1\right)$

$\approx €28\,200$

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Topic 2 | Graphs of functions

Theory card T Question cards

Front side

The **x-intercepts** of a function are the values of x for which $y = 0$. They are the **zeros** of the function.

The **y-intercept** of a function is the value of y when $x = 0$.

An **asymptote** is a line that the graph *approaches* or begins to look like as it tends to infinity in a particular direction.

- To find vertical asymptotes, look for values of x for which the function is undefined.
  - If $y = \frac{f(x)}{g(x)}$, find where $g(x) = 0$.
  - If $y = \log_a(f(x))$, find where $f(x) = 0$.
- To find horizontal asymptotes, consider the behaviour as $x \to \pm\infty$.

Unfold

Transformations of graphs

- $y = f(x) + b$ **translates** $y = f(x)$ vertically b units (upwards if $b > 0$).
- $y = f(x - a)$ **translates** $y = f(x)$ horizontally a units (right if $a > 0$).
- $y = f(x - a) + b$ **translates** $y = f(x)$ by the vector $\binom{a}{b}$.
- $y = pf(x)$, $p > 0$ is a **vertical stretch** of $y = f(x)$ with scale factor p.
- $y = f(qx)$, $q > 0$ is a **horizontal stretch** of $y = f(x)$ with scale factor $\frac{1}{q}$.
- $y = -f(x)$ is a **reflection** of $y = f(x)$ in the x-axis.
- $y = f(-x)$ is a **reflection** of $y = f(x)$ in the y-axis.
- If $f^{-1}(x)$ exists, $y = f^{-1}(x)$ is a **reflection** of $y = f(x)$ in the line $y = x$.

Invariant points are the points which do not move under a transformation.

Graphs of modulus functions

To graph $y = |f(x)|$, we reflect in the x-axis that part of the graph of $y = f(x)$ which lies below the x-axis; the rest remains invariant.

To graph $y = f(|x|)$, the part of the graph of $y = f(x)$ which lies to the right of the y-axis is invariant, and this part is reflected in the y-axis.

Graphs of reciprocal functions

- If $f(x) > 0$ then $\frac{1}{f(x)} > 0$.
- If $f(x) < 0$ then $\frac{1}{f(x)} < 0$.
- When $f(x)$ is a minimum, $\frac{1}{f(x)}$ is a maximum, and vice versa.
- Zeros of $f(x)$ correspond to vertical asymptotes of $\frac{1}{f(x)}$.
- Vertical asymptotes of $f(x)$ correspond to zeros of $\frac{1}{f(x)}$.
- Invariant points occur when $f(x) = \pm 1$.

Graphs of rational functions

You should be able to graph a rational function of the form $y = \frac{ax + b}{cx + d}$, $c \neq 0$ using transformations, and include its asymptotes.

Graphs of exponential and logarithmic functions

The graph of $y = a^x$ has the horizontal asymptote $y = 0$.

The graph of $y = \log_a x$ has the vertical asymptote $x = 0$.

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Question side

Consider the function $f(x) = e^x$.

a Find the equation of the resulting image when $y = f(x)$ is:

- **i** reflected in the x-axis
- **ii** translated by $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$
- **iii** stretched horizontally with scale factor $\frac{1}{4}$.

b Graph $y = f(x)$ and each of the transformations in **a** on the same set of axes.

Turn around

Answer side

a **i** For a reflection in the x-axis, $y = -f(x)$

$\therefore\ y = -e^x$

ii For a translation by $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $y = f(x-3) - 1$

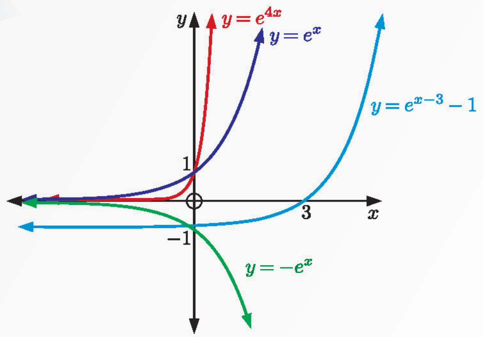
$\therefore\ y = e^{x-3} - 1$

iii For a horizontal stretch with scale factor $\frac{1}{4}$,

$y = f(4x)$

$\therefore\ y = e^{4x}$

b



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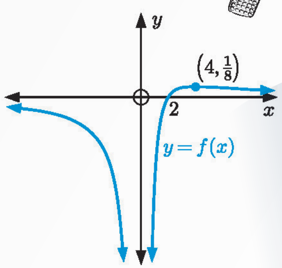
Question side

The graph alongside shows the function $f(x) = \dfrac{x-2}{x^2}$.



a Explain why $f(x)$ has no inverse function.

b State the asymptotes of $y = f(x)$.

c Suppose $g(x) = \dfrac{x-2}{x^2}$, $0 < x \leqslant 4$.

Draw, on the same set of axes:

$y = g(x)$, $y = g(-x)$, $y = -g(x)$, $y = \dfrac{1}{g(x)}$, $y = |g(x)|$,

$y = g(x+2)$, and $y = g^{-1}(x)$.

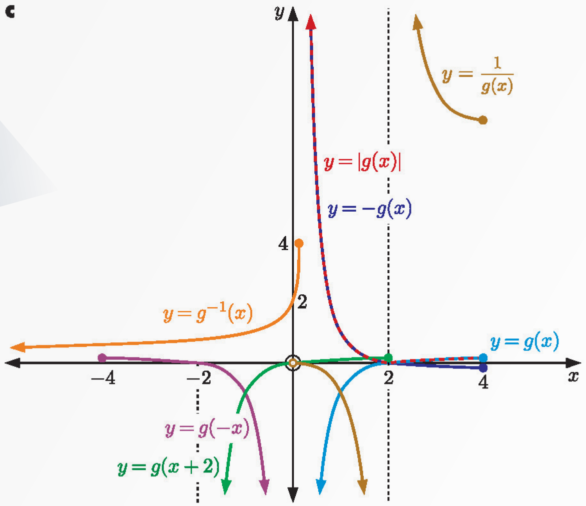
Turn around

Answer side

a $f(x)$ is not one-to-one, so it has no inverse.

b The asymptotes of $y = f(x)$ are $x = 0$ and $y = 0$.

c



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Topic 2 | Graphs of functions

Theory card | Question cards

Question side

44 | ? Topic 2 | Graphs of functions

Consider the function $f : x \mapsto 3\ln(4-x)$.

a State the domain and range of f.

b Find f^{-1}.

c Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Turn around

Answer side

44 | ! Topic 2 | Graphs of functions

a Domain $= \{x \mid x < 4\}$

Range $= \{y \mid y \in \mathbb{R}\}$

b f is $y = 3\ln(4-x)$

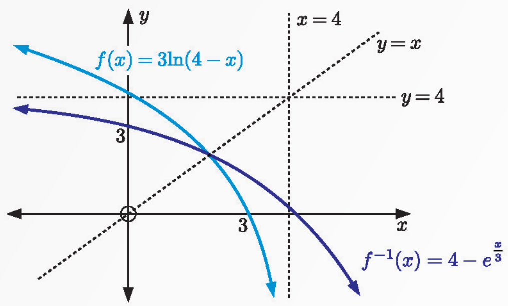
$\therefore\ f^{-1}$ is $x = 3\ln(4-y)$

$\therefore\ \frac{x}{3} = \ln(4-y)$

$\therefore\ e^{\frac{x}{3}} = 4-y$

$\therefore\ y = 4 - e^{\frac{x}{3}}$

c



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Question side

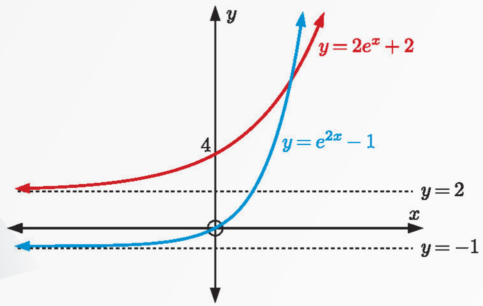
a Sketch the graphs of $y = e^{2x} - 1$ and $y = 2e^x + 2$ on the same set of axes.

b Find the coordinates of any points of intersection of the graphs.

Turn around

Answer side

a



b The graphs intersect when

$$e^{2x} - 1 = 2e^x + 2$$
$$\therefore \ e^{2x} - 2e^x - 3 = 0$$
$$\therefore \ (e^x - 3)(e^x + 1) = 0$$
$$\therefore \ e^x = 3 \quad \{\text{since } e^x > 0\}$$
$$\therefore \ x = \ln 3$$

When $x = \ln 3$, $y = 2e^{\ln 3} + 2 = 8$

$\therefore$ the graphs intersect at $(\ln 3, 8)$.

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Question side

Let $g(x) = \dfrac{3-x}{x+1}$.

a Sketch $y = g(x)$, clearly showing any axes intercepts and asymptotes.

Hence, sketch $y = \dfrac{1}{g(x)}$ on the same set of axes.

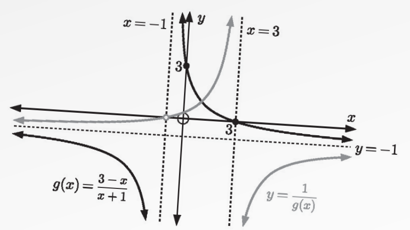
b Explain what happens to $y = \dfrac{1}{g(x)}$ around $x = -1$.

c State the position of any invariant points.

d Explain what transformations can be used to produce $y = g(x)$ from $y = \dfrac{1}{x}$.

Unfold

a



b Since $x = -1$ is an asymptote of $y = g(x)$, the point where $x = -1$ is not included on the graph of $y = \frac{1}{g(x)}$. However, the graph crosses the axis either side of this point.

c Invariant points occur when $\frac{1}{g(x)} = g(x)$

$$\therefore\ [g(x)]^2 = 1$$

$$\therefore\ \left(\frac{3-x}{x+1}\right)^2 = 1$$

$$\therefore\ (3-x)^2 = (x+1)^2$$

$$\therefore\ 3-x = \pm(x+1)$$

$$\therefore\ 3-x = x+1 \quad \text{or} \quad 3-x = -x-1$$

$$\therefore\ 2x = 2$$

$$\therefore\ x = 1$$

$g(1) = \frac{3-1}{1+1} = 1$

$\therefore$ the only invariant point is $(1, 1)$.

d $g(x) = \frac{3-x}{x+1} = \frac{-(x+1)+4}{x+1} = -1 + \frac{4}{x+1}$

So, if $y = g(x)$, then $y = \frac{4}{x+1} - 1$.

This is a translation by $\binom{-1}{-1}$ from $y = \frac{4}{x}$.

To obtain $y = g(x)$ from $y = \frac{1}{x}$, we first perform a vertical stretch of $y = \frac{1}{x}$ with scale factor 4. We then perform a translation of the resultant curve with translation vector $\binom{-1}{-1}$.