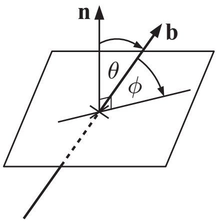
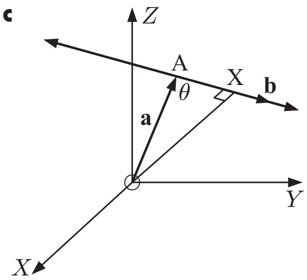
b 

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$\cos\theta = \sin\phi \quad \{\theta + \phi = 90°\}$

$$\therefore \quad \sin\phi = \frac{|\mathbf{n} \bullet \mathbf{b}|}{|\mathbf{n}|\,|\mathbf{b}|}$$
$$= \frac{|-2+1-9|}{\sqrt{4+1+9}\sqrt{1+1+9}}$$
$$= \frac{10}{\sqrt{14}\sqrt{11}}$$
$$= \frac{10}{\sqrt{154}}$$

$\therefore$ the angle between the line and the plane $\phi \approx 53.7°$.

c 

Let θ be the angle between $\overrightarrow{\text{OA}}$ and $\mathbf{b}$.

$$\therefore \quad \sin\theta = \frac{|\overrightarrow{\text{OX}}|}{|\mathbf{a}|}$$

Thus $|\overrightarrow{\text{OX}}| = |\mathbf{a}|\sin\theta$

Hence
$$|\overrightarrow{\text{OX}}|^2 = |\mathbf{a}|^2\sin^2\theta$$
$$= |\mathbf{a}|^2(1-\cos^2\theta)$$
$$= |\mathbf{a}|^2 - |\mathbf{a}|^2\cos^2\theta$$
$$= |\mathbf{a}|^2 - \frac{|\mathbf{a}|^2\,|\mathbf{b}|^2\cos^2\theta}{|\mathbf{b}|^2}$$
$$= |\mathbf{a}|^2 - \frac{(\mathbf{a}\bullet\mathbf{b})^2}{|\mathbf{b}|^2}$$
$$\therefore \quad |\overrightarrow{\text{OX}}| = \sqrt{|\mathbf{a}|^2 - \frac{(\mathbf{a}\bullet\mathbf{b})^2}{|\mathbf{b}|^2}}$$

d The line with equations $x = 2-t,\ y = 1+t,\ z = 1+3t,\ t \in \mathbb{R}$ contains the fixed point A(2, 1, 1) and has direction $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$.

Thus, as $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$,

$$|\overrightarrow{\text{OX}}| = \sqrt{(2^2+1^2+1^2) - \frac{(-2+1+3)^2}{((-1)^2+1^2+3^2)}}$$
$$= \sqrt{6 - \tfrac{4}{11}}$$
$$= \sqrt{\tfrac{62}{11}} \text{ units}$$

12 **a** **i**
$$\frac{P(x)}{(x-a)^2} = Q(x) + \frac{bx+c}{(x-a)^2}$$
$$\therefore \quad P(x) = Q(x)(x-a)^2 + bx + c$$

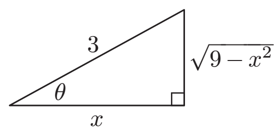
ii
$$P(a) = Q(a)\times 0 + ab + c$$
$$\therefore \quad P(a) = ab + c$$
$$P'(x) = Q'(x)(x-a)^2 + Q(x)2(x-a) + b$$
$$\therefore \quad P'(a) = 0 + 0 + b$$
$$= b$$

iii Remainder $= bx + c$
$$= P'(a)x + (P(a) - ab)$$
$$= P'(a)x + P(a) - aP'(a)$$
$$= P'(a)(x-a) + P(a)$$

iv For $P(x) = x^5$ divided by $(x+2)^2$,
$$P(-2) = -32$$
and $P'(x) = 5x^4$
$$\therefore \quad P'(-2) = 5 \times 16$$
$$= 80$$
$$\therefore \quad \text{remainder} = 80(x+2) - 32$$
$$= 80x + 128$$

b Let $x = 3\cos\theta$ so $dx = -3\sin\theta\, d\theta$

$$\therefore \quad \int \frac{x}{\sqrt{9-x^2}}\,dx$$
$$= \int \frac{3\cos\theta}{\sqrt{9-9\cos^2\theta}}(-3\sin\theta)\,d\theta$$
$$= \int \frac{3\cos\theta}{3\sin\theta}(-3\sin\theta)\,d\theta$$
$$= \int -3\cos\theta\,d\theta$$
$$= -3\sin\theta + c$$
$$= -3\left(\frac{\sqrt{9-x^2}}{3}\right) + c$$
$$= -\sqrt{9-x^2} + c$$



Check:
$$\frac{d}{dx}(-(9-x^2)^{\frac{1}{2}} + c)$$
$$= -\tfrac{1}{2}(9-x^2)^{-\frac{1}{2}}(-2x) + 0$$
$$= \frac{x}{\sqrt{9-x^2}} \quad ✓$$

SOLUTIONS TO TRIAL EXAMINATION 3

NO CALCULATOR

SECTION A

1 **a**
$$\frac{p(x)}{x(2x-3)} = ax + b + \frac{ax+b}{x(2x-3)}$$
$$\therefore \quad p(x) = x(ax+b)(2x-3) + ax + b$$
$$\therefore \quad p(x) = (ax+b)[x(2x-3)+1]$$

b Thus $p(x) = (ax+b)(2x^2-3x+1)$
$$= (ax+b)(2x-1)(x-1)$$
$\Rightarrow \quad (2x-1)$ and $(x-1)$ are factors of $p(x)$.

c Since $p(0) = 7$ and $p(2) = 39$,
$b(-1)(-1) = 7$ and $(2a+b)(3)(1) = 39$
$\therefore \quad b = 7$ and $2a + 7 = 13$
$\therefore \quad 2a = 6$
$\therefore \quad a = 3$

Thus, $p(x) = (3x+7)(2x^2-3x+1)$
$$= 6x^3 + 5x^2 - 18x + 7$$

2 **a** Since $\ln\left(\dfrac{a^2}{b}\right) = k,\ \dfrac{a^2}{b} = e^k$.

Likewise, since $\ln\left(\dfrac{b^2}{a^3}\right) = 2,\ \dfrac{b^2}{a^3} = e^2$.

Thus $a^2 = be^k$ and $a^3 = \dfrac{b^2}{e^2}$

$\therefore \quad a^6 = b^3e^{3k}$ and $a^6 = \dfrac{b^4}{e^4}$

$\therefore \quad \dfrac{b^4}{e^4} = b^3e^{3k} \quad$ {values for a^6}

$\therefore \quad b = e^{3k+4}$

b Since $\dfrac{b^2}{a^3} = e^2, \quad a^3 = \dfrac{b^2}{e^2}$

$$\therefore \quad a^3 = \frac{e^{6k+8}}{e^2}$$
$$\therefore \quad a^3 = e^{6k+6}$$
$$\therefore \quad a = \left(e^{6k+6}\right)^{\frac{1}{3}}$$
$$\therefore \quad a = e^{2k+2}, \quad \text{so } r = 2 \text{ and } s = 2.$$

3 **a** Let $\dfrac{x-2}{x^2-1} = \dfrac{a}{x+1} + \dfrac{b}{x-1}$

$$= \frac{a(x-1)+b(x+1)}{(x+1)(x-1)}$$
$$= \frac{(a+b)x+(b-a)}{x^2-1}$$

$\therefore \quad a+b = 1$ and $b-a = -2$ {equating coefficients}

Solving these equations simultaneously gives $b = -\frac{1}{2}$ and $a = \frac{3}{2}$.

b
$$\int_{-4}^{-2} \frac{x-2}{x^2-1}\,dx$$
$$= \int_{-4}^{-2} \left(\frac{\frac{3}{2}}{x+1} - \frac{\frac{1}{2}}{x-1}\right) dx$$
$$= \left[\tfrac{3}{2}\ln|x+1| - \tfrac{1}{2}\ln|x-1|\right]_{-4}^{-2}$$
$$= (\tfrac{3}{2}\ln 1 - \tfrac{1}{2}\ln 3) - (\tfrac{3}{2}\ln 3 - \tfrac{1}{2}\ln 5)$$
$$= 0 - \tfrac{1}{2}\ln 3 - \tfrac{3}{2}\ln 3 + \tfrac{1}{2}\ln 5$$
$$= -2\ln 3 + \tfrac{1}{2}\ln 5$$

c $\dfrac{x-2}{x^2-1}$ is undefined when $x = \pm 1$ and $x = 1$ lies in the domain of integration $[1, 3]$.

Thus $\displaystyle\int_1^3 \frac{x-2}{x^2-1}\,dx$ does not exist.

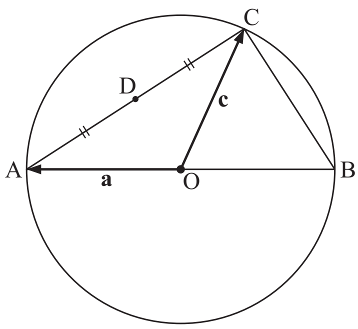
4 **a** Let $u = x \qquad v' = \sin 2x$

$u' = 1 \qquad v = -\frac{1}{2}\cos 2x$

$$\therefore \quad \int x\sin 2x\,dx$$
$$= uv - \int u'v\,dx$$
$$= -\frac{x}{2}\cos 2x - \int -\tfrac{1}{2}\cos 2x\,dx$$
$$= -\frac{x}{2}\cos 2x + \tfrac{1}{2}(\tfrac{1}{2}\sin 2x) + c$$
$$= \tfrac{1}{4}\sin 2x - \frac{x}{2}\cos 2x + c$$

b
$$A = \int_0^{\frac{\pi}{2}} x\sin 2x\,dx$$
$$= \left[\tfrac{1}{4}\sin 2x - \frac{x}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= (\tfrac{1}{4}\sin\pi - \tfrac{\pi}{4}\cos\pi) - (\tfrac{1}{4}\sin 0 - 0)$$
$$= 0 - \tfrac{\pi}{4}(-1) - 0$$
$$= \tfrac{\pi}{4} \text{ units}^2$$
$$B = -\int_{\frac{\pi}{2}}^{\pi} x\sin 2x\,dx$$
$$= -\left[\tfrac{1}{4}\sin 2x - \frac{x}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\pi}$$
$$= -\left[(\tfrac{1}{4}\sin 2\pi - \tfrac{\pi}{2}\cos 2\pi) - (\tfrac{1}{4}\sin\pi - \tfrac{\pi}{4}\cos\pi)\right]$$
$$= 0 + \tfrac{\pi}{2} - 0 + \tfrac{\pi}{4}$$
$$= \tfrac{3\pi}{4} \text{ units}^2$$
$$\therefore \quad B = 3A$$

5 **a**



$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$$
$$= -\mathbf{a} + \mathbf{c}$$
$$= \mathbf{c} - \mathbf{a}$$
$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$
$$= \mathbf{a} + \tfrac{1}{2}\overrightarrow{AC}$$
$$= \mathbf{a} + \tfrac{1}{2}(\mathbf{c} - \mathbf{a})$$
$$= \mathbf{a} + \tfrac{1}{2}\mathbf{c} - \tfrac{1}{2}\mathbf{a}$$
$$= \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{c}$$

b
$$\overrightarrow{AC} \bullet \overrightarrow{OD} = (\mathbf{c} - \mathbf{a}) \bullet (\tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{c})$$
$$= \tfrac{1}{2}\mathbf{a} \bullet \mathbf{c} + \tfrac{1}{2}\mathbf{c} \bullet \mathbf{c} - \tfrac{1}{2}\mathbf{a} \bullet \mathbf{a} - \tfrac{1}{2}\mathbf{a} \bullet \mathbf{c}$$
$$= \tfrac{1}{2}\left(|\mathbf{c}|^2 - |\mathbf{a}|^2\right)$$
$$= 0 \quad \text{as} \quad |\mathbf{a}| = |\mathbf{c}| \quad \text{\{equal radii\}}$$

$\therefore$ $\overrightarrow{AC}$ and $\overrightarrow{OD}$ are perpendicular

$\therefore$ [AC] $\perp$ [OD].

c We have just proved that "the line from the centre of a circle to the midpoint of a chord, is perpendicular to the chord".

6 **a** $g(x)$ is defined when $3 - 2x \geqslant 0$

$$\therefore \quad -2x \geqslant -3$$
$$\therefore \quad x \leqslant \tfrac{3}{2}$$

b
$$(g \circ f)(x) = g(f(x))$$
$$= g(x^2 + 4x)$$
$$= \sqrt{3 - 2(x^2 + 4x)}$$
$$\therefore \quad (g \circ f)(-3) = \sqrt{3 - 2(9 - 12)}$$
$$= \sqrt{3 - 2(-3)}$$
$$= 3$$

c $f(x) = x^2 + 4x$ where $x \leqslant -2$ has inverse y where $x = y^2 + 4y$ where $y \leqslant -2$

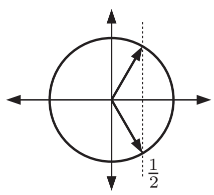
$$\Rightarrow \quad y^2 + 4y - x = 0, \; y \leqslant -2$$
$$\Rightarrow \quad y = \frac{-4 \pm \sqrt{16 - 4(1)(-x)}}{2}, \; y \leqslant -2$$
$$\Rightarrow \quad y = \frac{-4 \pm \sqrt{4(4+x)}}{2}, \; y \leqslant -2$$
$$\Rightarrow \quad y = -2 \pm \sqrt{4+x}, \; y \leqslant -2$$
$$\Rightarrow \quad y = -2 - \sqrt{4+x}$$
$$\therefore \quad f^{-1}(x) = -2 - \sqrt{4+x}, \; x \geqslant -4$$

7 **a** $2^{2x} + 2^{x+1} = 15$

Let $m = 2^x$

$$\therefore \quad m^2 + 2m - 15 = 0$$
$$\therefore \quad (m-3)(m+5) = 0$$
$$\therefore \quad m = 3 \text{ or } -5$$
$$\therefore \quad 2^x = 3 \qquad \{2^x > 0 \text{ for all } x\}$$
$$\therefore \quad x = \log_2 3$$

b
$$\sin^2 x + \cos x = 1.25$$
$$\therefore \quad 1 - \cos^2 x + \cos x - 1\tfrac{1}{4} = 0$$
$$\therefore \quad \cos^2 x - \cos x + \tfrac{1}{4} = 0$$
$$\therefore \quad 4\cos^2 x - 4\cos x + 1 = 0$$
$$\therefore \quad (2\cos x - 1)^2 = 0$$
$$\therefore \quad \cos x = \tfrac{1}{2}$$
$$\therefore \quad x = \pm\tfrac{\pi}{3}$$



8 $f(x) = \frac{1}{2}\cos 2x + \sin x$

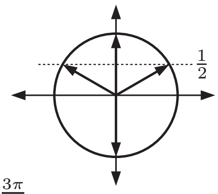
a $f(0) = \frac{1}{2}\cos 0 + \sin 0 = \frac{1}{2}$ $\quad\therefore$ A is $(0,\ \frac{1}{2})$

$f(2\pi) = \frac{1}{2}\cos 4\pi + \sin 2\pi = \frac{1}{2}$ $\quad\therefore$ H is $(2\pi,\ \frac{1}{2})$

b $f'(x) = \frac{1}{2}(-2\sin 2x) + \cos x$

$\quad = -2\sin x\cos x + \cos x$

$\quad = \cos x(1 - 2\sin x)$



$\therefore\ f'(x) = 0 \ \Leftrightarrow\ \cos x = 0$

$\qquad\qquad$ or $\sin x = \frac{1}{2}$

$\qquad \Leftrightarrow\ x = \frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{3\pi}{2}$

$\therefore$ B is $(\frac{\pi}{6},\ \frac{3}{4})$, C is $(\frac{\pi}{2},\ \frac{1}{2})$, D is $(\frac{5\pi}{6},\ \frac{3}{4})$, and F is $(\frac{3\pi}{2},\ -\frac{3}{2})$.

c The x-coordinates of E and G are the solutions of $\frac{1}{2}\cos 2x + \sin x = 0$

$\therefore\ \cos 2x + 2\sin x = 0$

$\therefore\ 1 - 2\sin^2 x + 2\sin x = 0$

$\therefore\ 2\sin^2 x - 2\sin x - 1 = 0$

$$\therefore\ \sin x = \frac{2 \pm \sqrt{4 - 4(2)(-1)}}{4}$$

$$\therefore\ \sin x = \frac{2 \pm 2\sqrt{3}}{4} = \frac{1 \pm \sqrt{3}}{2}$$

But $-1 \leqslant \sin x \leqslant 1$, so $\sin x = \dfrac{1 - \sqrt{3}}{2}$.

At E, $x = \pi - \arcsin\left(\dfrac{1-\sqrt{3}}{2}\right)$.

At G, $x = \arcsin\left(\dfrac{1-\sqrt{3}}{2}\right) + 2\pi$.

SECTION B

9 **a** **i** $\cos(A + B) + \cos(A - B)$

$= \cos A\cos B - \cancel{\sin A\sin B}$

$\qquad + \cos A\cos B + \cancel{\sin A\sin B}$

$= 2\cos A\cos B$

ii $\sin(A + B) + \sin(A - B)$

$= \sin A\cos B + \cancel{\cos A\sin B}$

$\qquad + \sin A\cos B - \cancel{\cos A\sin B}$

$= 2\sin A\cos B$

b **i** Using **a i** with $A = 2\theta,\ B = \theta$

$\therefore\ \cos(2\theta + \theta) + \cos(2\theta - \theta) = 2\cos 2\theta\cos\theta$

$\therefore\ \cos 3\theta + \cos\theta = 2\cos 2\theta\cos\theta$

ii Likewise using **a ii** with $A = 2\theta,\ B = \theta$

$\sin 3\theta + \sin\theta = 2\sin 2\theta\cos\theta$

c $z^2 = [\text{cis}\,\theta]^2 = \text{cis}\,2\theta$

and $z^3 = [\text{cis}\,\theta]^3 = \text{cis}\,3\theta$ $\quad$ {De Moivre's theorem}

d $z^3 + z = \text{cis}\,3\theta + \text{cis}\,\theta$

$= \cos 3\theta + i\sin 3\theta + \cos\theta + i\sin\theta$

$= [\cos 3\theta + \cos\theta] + i[\sin 3\theta + \sin\theta]$

$= 2\cos 2\theta\cos\theta + i2\sin 2\theta\cos\theta$

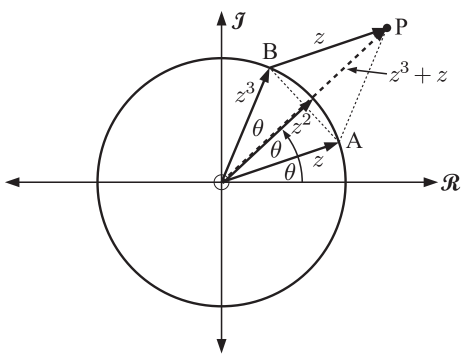
$= 2\cos\theta[\cos 2\theta + i\sin 2\theta]$

$= 2\cos\theta\,\text{cis}\,2\theta$

e From **d**, $\left|z^3 + z\right| = 2\cos\theta$ and

$\arg(z^3 + z) = 2\theta$

f



$z^2 = \text{cis}\,2\theta$

$\therefore\ \arg(z^2) = 2\theta$

$\therefore\ \arg(z^3 + z) = \arg(z^2)$

g If $z + z^3$ is purely imaginary, then $\overrightarrow{\text{OP}}$ lies on the imaginary axis

$\therefore\ 2\theta = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

$\therefore\ \theta = \frac{\pi}{4} + k\frac{\pi}{2},\ k \in \mathbb{Z}$

$\therefore\ \theta = \pm\frac{\pi}{4}$ $\qquad \{-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}\}$

10 **a** **i** $f(x) = \tan^3 x$

$\therefore\ f'(x) = 3\tan^2 x \times \sec^2 x$

$= 3(\sec^2 x - 1)\sec^2 x$

$= 3\sec^4 x - 3\sec^2 x$

ii Using **i**,

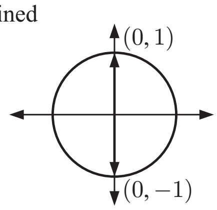
$\int (3\sec^4 x - 3\sec^2 x)\,dx = \tan^3 x + c$

$\therefore\ 3\int \sec^4 x\,dx - 3\int \sec^2 x\,dx = \tan^3 x + c$

$\therefore\ 3\int \sec^4 x\,dx - 3\tan x = \tan^3 x + c$

$\therefore\ \int \sec^4 x\,dx = \tan x + \frac{1}{3}\tan^3 x + c$

b **i** $y = \sec^2 x = \dfrac{1}{\cos^2 x}$ is undefined when $\cos x = 0$



$\Rightarrow$ for the illustrated graph, $a = \frac{\pi}{2}$

ii When $x = \frac{\pi}{3}$, $\cos x = \frac{1}{2}$

$\therefore\ \sec^2 x = 4$

$\therefore$ A is $(\frac{\pi}{3},\ 4)$

c Shaded area $= \displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$

$= \Big[\tan x\Big]_0^{\frac{\pi}{3}}$

$= \tan\frac{\pi}{3} - \tan 0$

$= \sqrt{3}$ units^2

d Volume $= \pi\displaystyle\int_0^{\frac{\pi}{3}} y^2\,dx$

$= \pi\displaystyle\int_0^{\frac{\pi}{3}} \sec^4 x\,dx$

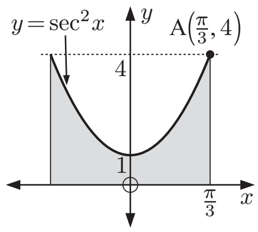
$= \pi\left[\tan x + \frac{1}{3}\tan^3 x\right]_0^{\frac{\pi}{3}}$ $\qquad$ {using **a**}

$= \pi(\sqrt{3} + \frac{1}{3}(\sqrt{3})^3 - 0)$

$= \pi(\sqrt{3} + \sqrt{3})$

$= 2\pi\sqrt{3}$ units^3

e



$$y = \sec^2 x$$
$$\therefore\ \sec x = \sqrt{y} \quad \{\sec x > 0\}$$
$$\therefore\ \cos x = \frac{1}{\sqrt{y}}$$
$$\therefore\ x = \arccos\left(\frac{1}{\sqrt{y}}\right)$$

Volume of cylinder $= \pi\left(\frac{\pi}{3}\right)^2 4$

$= \frac{4\pi^3}{9}$

$\therefore$ volume required $= \frac{4\pi^3}{9} - \pi\int_1^4 x^2\,dy$

$= \frac{4\pi^3}{9} - \pi\int_1^4 \left[\arccos\left(\frac{1}{\sqrt{y}}\right)\right]^2 dy$

11 **a** $\mathbf{r} = \begin{pmatrix} 2t+5 \\ -2t-1 \\ t \end{pmatrix}$ is the equation of L.

When $t = 2$, $\mathbf{r} = \begin{pmatrix} 9 \\ -5 \\ 2 \end{pmatrix}$ $\therefore$ $(9, -5, 2)$ lies on L.

b $\mathbf{n} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$ and the plane has equation

$3x - 4y - z = 3(-1) - 4(0) - (4)$

which is $3x - 4y - z = -7$

c The line $x = 2t+5$, $y = -2t-1$, $z = t$ meets the plane when $3(2t+5) - 4(-2t-1) - t = -7$

$\therefore\ 6t + 15 + 8t + 4 - t = -7$

$\therefore\ 13t = -26$

$\therefore\ t = -2$

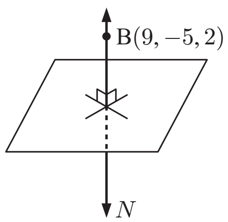
Hence $x = 2(-2) + 5 = 1$,

$y = -2(-2) - 1 = 3$,

$z = -2$

The line meets the plane at $(1, 3, -2)$.

d



Line N has equation

$x = 9 + 3s$, $y = -5 - 4s$,

$z = 2 - s$, $s \in \mathbb{R}$.

or $\mathbf{r}_2 = \begin{pmatrix} 9+3s \\ -5-4s \\ 2-s \end{pmatrix}$

e Line N meets P where

$3(9+3s) - 4(-5-4s) - (2-s) = -7$

$\therefore\ 27 + 9s + 20 + 16s - 2 + s = -7$

$\therefore\ 26s + 45 = -7$

$\therefore\ 26s = -52$

$\therefore\ s = -2$

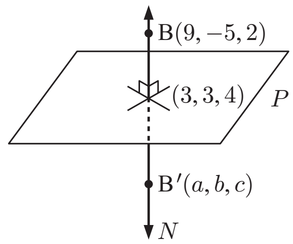
Hence $x = 9 + 3(-2) = 3$

$y = -5 - 4(-2) = 3$

$z = 2 - (-2) = 4$

$\therefore$ line N meets the plane at $(3, 3, 4)$.

f



If B′ is (a, b, c),

$\frac{a+9}{2} = 3,\quad \frac{b-5}{2} = 3,\quad \frac{c+2}{2} = 4$

$\therefore\ a + 9 = 6,\quad b - 5 = 6,\quad c + 2 = 8$

$\therefore\ a = -3,\quad b = 11,\quad c = 6$

$\therefore$ B′ is $(-3, 11, 6)$

g Given $\text{C}(1, 3, -2)$ and $\text{B}'(-3, 11, 6)$,

$\overrightarrow{\text{CB}'} = \begin{pmatrix} -4 \\ 8 \\ 8 \end{pmatrix} = -4(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$

$\Rightarrow$ $\overrightarrow{\text{CB}'}$ is parallel to $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

{$\overrightarrow{\text{CB}'}$ is a constant multiple of $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$}

12 **a** P_n is: "$2^{4n+3} + 3^{3n+1}$ is divisible by 11" for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $2^{4n+3} + 3^{3n+1} = 2^7 + 3^4$

$= 128 + 81$

$= 209$

$= 11 \times 19$

$\therefore$ P_1 is true.

(2) If P_k is true then

$2^{4k+3} + 3^{3k+1} = 11A$ where $A \in \mathbb{Z}^+$.

Now $2^{4(k+1)+3} + 3^{3(k+1)+1}$

$= 2^{4k+4+3} + 3^{3k+3+1}$

$= 2^4 2^{4k+3} + 3^3 3^{3k+1}$

$= 16(11A - 3^{3k+1}) + 27 \times 3^{3k+1}$

$= 16 \times 11 \times A - 16 \times 3^{3k+1} + 27 \times 3^{3k+1}$

$= 16 \times 11 \times A + 11 \times 3^{3k+1}$

$= 11(16A + 3^{3k+1})$ where $16A + 3^{3k+1} \in \mathbb{Z}^+$

Thus P_{k+1} is true.

So, P_1 is true, and P_{k+1} is true provided P_k is true for all $k \geqslant 1$.

Hence, P_n is true.

{Principle of mathematical induction}

b Since a, b, c are consecutive terms in an arithmetic sequence, $b - a = c - b$

$\therefore\ a + c = 2b$

But $a + b + c = 33$, so $2b + b = 33$

$\therefore\ 3b = 33$

$\therefore\ b = 11$

Since a, $b+1$, $c+29$ are consecutive terms in a geometric sequence,

$\frac{b+1}{a} = \frac{c+29}{b+1}$

$\therefore\ \frac{12}{a} = \frac{c+29}{12}$ where $a + c = 22$

Thus $\dfrac{12}{a} = \dfrac{22 - a + 29}{12}$

$\therefore \ \dfrac{12}{a} = \dfrac{51 - a}{12}$

$\therefore \ 144 = 51a - a^2$

$\therefore \ a^2 - 51a + 144 = 0$

$\therefore \ (a-3)(a-48) = 0$

$\therefore \ a = 3 \text{ or } 48$

$\therefore \ a = 3, \quad b = 11, \ c = 19$ or
$a = 48, \ b = 11, \ c = -26$

CALCULATOR

SECTION A

1 **a** **i** $z = r \operatorname{cis} \theta$

$z^3 = (r \operatorname{cis} \theta)^3$
$= r^3 \operatorname{cis} 3\theta$ {De Moivre's theorem}

ii $\sqrt[3]{z} = z^{\frac{1}{3}}$
$= (r \operatorname{cis} \theta)^{\frac{1}{3}}$
$= r^{\frac{1}{3}} \operatorname{cis} \left(\frac{\theta}{3}\right)$ {De Moivre's theorem}

b $-11 + ai = (1 - ai)^3$
$= 1 - 3(ai) + 3(ai)^2 - (ai)^3$
$= 1 - 3ai - 3a^2 + a^3 i$
$= (1 - 3a^2) + (a^3 - 3a)i$

$\Rightarrow \ 1 - 3a^2 = -11$ and $a^3 - 3a = a$
{equating real and imaginary parts}

$\therefore \ 3a^2 = 12$ **and** $a^3 - 4a = 0$

$\therefore \ a^2 = 4$ **and** $a(a^2 - 4) = 0$

$\therefore \ a = \pm 2$

2 **a** If $f(x) = \begin{cases} \frac{1}{2}e^{-bx}, & 0 \leqslant x \leqslant 1, \quad b \neq 0 \\ 0, & \text{otherwise} \end{cases}$

is a well defined PDF, then

$$\int_0^1 \tfrac{1}{2}e^{-bx}\, dx = 1 \qquad \{f(x) > 0 \ \text{always}\}$$

$$\therefore \ \left[\frac{1}{-2b}e^{-bx}\right]_0^1 = 1$$

$\therefore \ e^{-b} - e^0 = -2b$

$\therefore \ e^{-b} + 2b - 1 = 0$

b Using technology, $b \approx -1.256\,431$
$\therefore \ b \approx -1.256$

c $\mu = \text{E}(X)$

$$\approx \int_0^1 x\left(\tfrac{1}{2}e^{1.256\,431x}\right) dx$$

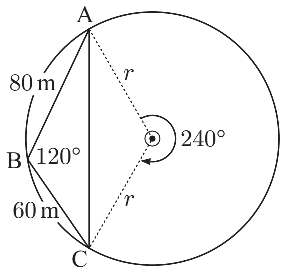
≈ 0.602 {using technology}

d $\text{Var}(X)$
$= \text{E}(X^2) - \{\text{E}(X)\}^2$

$$\approx \int_0^1 x^2\left(\tfrac{1}{2}e^{1.256\,431x}\right) dx - \mu^2$$

≈ 0.0772 {using technology}

3 **a**



Using the cosine rule in $\triangle$ABC,

$\text{AC}^2 = 60^2 + 80^2 - 2(60)(80)\cos 120°$

$\therefore \ \text{AC}^2 = 60^2 + 80^2 + 60 \times 80$

$\therefore \ \text{AC}^2 = 14\,800$

b Reflex $\text{A}\widehat{\text{O}}\text{C} = 240°$ {angle at the centre theorem}

$\therefore \ \text{A}\widehat{\text{O}}\text{C} = 120°$

In $\triangle$AOC, $\text{AC}^2 = r^2 + r^2 - 2rr\cos 120°$ {cosine rule}

$\therefore \ \text{AC}^2 = 2r^2 + r^2$

$\therefore \ \text{AC}^2 = 3r^2$

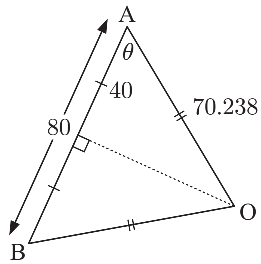
c From **a** and **b**, $3r^2 = 14\,800$

$\therefore \ r^2 \approx 4933.33....$

$\therefore \ r \approx 70.238....$

$\therefore \ r \approx 70.2$

d



$\cos\theta \approx \dfrac{40}{70.238}$

$\theta \approx \arccos\left(\dfrac{40}{70.238}\right)$

$\therefore \ \text{B}\widehat{\text{A}}\text{O} \approx 55.3°$

4 **a** When no replacement occurs, we do not have a repetition of n independent trials each with the same probability of success. However, since n is very large, the probability of a success each time will be almost the same.

b $X \sim \text{B}(20, 0.032)$

i $\text{P}(X \leqslant 2)$
$= \text{P}(X = 0, 1, \text{ or } 2)$
$= \binom{20}{0}(0.032)^0(0.968)^{20} + \binom{20}{1}(0.031)^1(0.968)^{19}$
$+ \binom{20}{2}(0.032)^2(0.968)^{18}$
≈ 0.975

ii $\text{P}(X \geqslant 4)$
$= 1 - \text{P}(X \leqslant 3)$
$\approx 0.003\,37$ {using technology}

5 **a** $\dfrac{d}{dx}(x^{n+1}\ln x) = (n+1)x^n \ln x + x^{n+1}\left(\dfrac{1}{x}\right)$
$= (n+1)x^n \ln x + x^n$

Thus $\int [(n+1)x^n \ln x + x^n]\, dx = x^{n+1}\ln x + c$

$\therefore \ (n+1)\int x^n \ln x\, dx + \dfrac{x^{n+1}}{n+1} = x^{n+1}\ln x + c$
provided $n \neq -1$

$\therefore \ \int x^n \ln x\, dx = \dfrac{x^{n+1}\ln x}{n+1} - \dfrac{x^{n+1}}{(n+1)^2} + c$

$= \dfrac{x^{n+1}}{(n+1)^2}\left((n+1)\ln x - 1\right) + c$
provided $n \neq -1$

b When $n = -1$,

$$\int x^n \ln x\, dx = \int \frac{\ln x}{x}\, dx$$
$$= \int (\ln x)^1 \left(\frac{1}{x}\right) dx$$

This has the form $\int (f(x))^n f'(x)\, dx$, so

$$\int x^n \ln x\, dx = \frac{(\ln x)^2}{2} + c$$

6 $s = t\sin\left(\frac{t}{2}\right) + 2\cos\left(\frac{t}{2}\right)$ metres

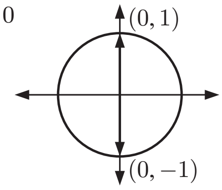
a $v = \frac{ds}{dt} = (1)\sin\left(\frac{t}{2}\right) + t\left(\frac{1}{2}\right)\cos\left(\frac{t}{2}\right) + 2\left(\frac{1}{2}\right)\left(-\sin\left(\frac{t}{2}\right)\right)$

$\therefore\ v = \cancel{\sin\left(\frac{t}{2}\right)} + \frac{t}{2}\cos\left(\frac{t}{2}\right) - \cancel{\sin\left(\frac{t}{2}\right)}$

$\therefore\ v = \frac{t}{2}\cos\left(\frac{t}{2}\right)\ \text{m s}^{-1}$

b The particle is at rest when $v = 0$

$\therefore\ \frac{t}{2}\cos\left(\frac{t}{2}\right) = 0$



$\therefore\ t = 0$ or $\frac{t}{2} = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

$\therefore\ t = 0$ or $t = \pi + k2\pi$

At $t = \pi$, $\cos\left(\frac{t}{2}\right)$ changes sign.

$\therefore$ the particle reverses direction at $t = \pi$ seconds.

Its position at this time is $s(\pi) = \pi$ m right of O.

c $a = \frac{dv}{dt} = \frac{1}{2}\cos\left(\frac{t}{2}\right) + \frac{t}{2}\left(-\sin\left(\frac{t}{2}\right)\right)\frac{1}{2}$

$\therefore\ a(t) = \frac{1}{2}\cos\left(\frac{t}{2}\right) - \frac{t}{4}\sin\left(\frac{t}{2}\right)$

$\therefore\ a\left(\frac{\pi}{3}\right) = \frac{1}{2}\cos\left(\frac{\pi}{6}\right) - \frac{\pi}{12}\sin\left(\frac{\pi}{6}\right)$

$= \frac{1}{2} \times \frac{\sqrt{3}}{2} - \frac{\pi}{12} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{4} - \frac{\pi}{24}\ \text{m s}^{-2}$

7 **a** $\arctan\left(\frac{x}{3}\right) + \arctan 6 = \arctan 3$

Let $\alpha \quad + \quad \beta \quad = \quad \theta$

where $\tan\alpha = \frac{x}{3}$, $\tan\beta = 6$, $\tan\theta = 3$.

Now $\tan(\alpha + \beta) = \tan\theta$

$\therefore\ \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan\theta$

$\therefore\ \frac{\frac{x}{3} + 6}{1 - \frac{x}{3}(6)} = 3$

$\therefore\ \frac{x}{3} + 6 = 3(1 - 2x)$

$\therefore\ x + 18 = 9 - 18x$

$\therefore\ 19x = -9$

$\therefore\ x = -\frac{9}{19}$

b $y = \arctan\left(\frac{x}{3}\right)$

$\therefore\ \frac{dy}{dx} = \frac{1}{1 + \left(\frac{x}{3}\right)^2} \times \frac{1}{3}$

$= \frac{1}{1 + \frac{x^2}{9}} \times \frac{1}{3}$

$= \frac{3}{9 + x^2}$

8 **a** If $x = 0$, $y = 3$, $\quad 9 + 3b + c = 0$ .... (1)

If $x = 2$, $y = -1$, $\quad 5 + 2a - b + c = 0$ .... (2)

If $x = 8$, $y = 7$, $\quad 113 + 8a + 7b + c = 0$ .... (3)

b
$$\left[\begin{array}{ccc|c} 8 & 7 & 1 & -113 \\ 2 & -1 & 1 & -5 \\ 0 & 3 & 1 & -9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 8 & 7 & 1 & -113 \\ 0 & -11 & 3 & 93 \\ 0 & 3 & 1 & -9 \end{array}\right] \quad R_2 \to 4R_2 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 8 & 7 & 1 & -113 \\ 0 & 3 & 1 & -9 \\ 0 & 0 & 20 & 180 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_3 \\ R_3 \to 3R_2 + 11R_3 \end{array}$$

c From row 3, $20c = 180 \quad \therefore\ c = 9$

From row 2, $3b + 9 = -9 \quad \therefore\ 3b = -18$

$\therefore\ b = -6$

From row 1, $8a + 7(-6) + 9 = -113$

$\therefore\ 8a - 42 + 9 = -113$

$\therefore\ 8a = -80$

$\therefore\ a = -10$

$\therefore\ a = -10$, $b = -6$, and $c = 9$

SECTION B

9 Let X = result of a student in the Science exam.

$X \sim \text{N}(56.7,\ 18.2^2)$

a **i** P(result between 65 and 85 inclusive)

$= \text{P}(65 \leqslant X \leqslant 85)$

≈ 0.264

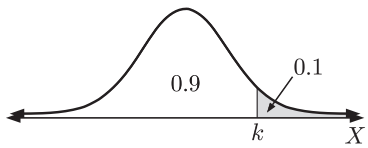
ii P(result of at least 70)

$= \text{P}(X \geqslant 70)$

≈ 0.232

b **i** We need to find k where

$\text{P}(X \leqslant k) = 0.9$

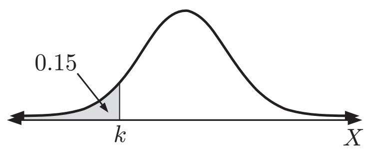


$\therefore\ k \approx 80$

$\therefore$ a result of 80 or more receives an 'A'.

ii We need to find k where

$\text{P}(X \leqslant k) = 0.15$



$\therefore\ k \approx 37.8$

$\therefore$ a result of 37 or less receives an 'F'.

c

Student	(1)	(2)	(3)	(4)
	A	A	F	F
	A	F	A	F
	A	F	F	A
	F	F	A	A
	F	A	F	A
	F	A	A	F

There are six different permutations of 2 'A's and 2 'F's.

$\therefore$ P(two 'A's and two 'F's)

$= 6 \times (0.1)^2 \times (0.15)^2$

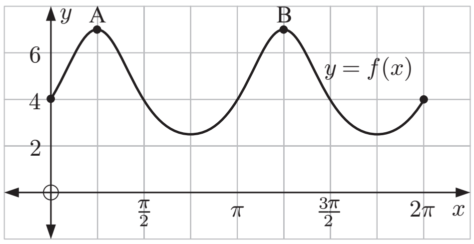
$= 0.001\,35$

d $Y \sim \text{B}(20,\ 0.1)$

i $\text{P}(Y=0) \approx 0.122$

ii $\text{P}(Y \geqslant 3) = 1 - \text{P}(Y \leqslant 2) \approx 0.323$

10 **a** **i** $f(x) = \dfrac{12 + 2\sin 2x}{3 - \sin 2x}$

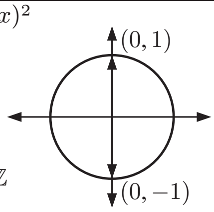


ii $f'(x)$

$$= \frac{4\cos 2x(3-\sin 2x) - (12 + 2\sin 2x)(-2\cos 2x)}{(3-\sin 2x)^2}$$

$$= \frac{12\cos 2x - \cancel{4\sin 2x\cos 2x} + 24\cos 2x + \cancel{4\sin 2x\cos 2x}}{(3-\sin 2x)^2}$$

$$= \frac{36\cos 2x}{(3-\sin 2x)^2}$$



$\therefore\ f'(x) = 0$

$\Leftrightarrow\ \cos 2x = 0$

$\Leftrightarrow\ 2x = \frac{\pi}{2} + k\pi,\ \ k \in \mathbb{Z}$

$\Leftrightarrow\ x = \frac{\pi}{4} + k\frac{\pi}{2}$

$\Leftrightarrow\ x = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$

$\therefore$ the local maximum at A is $\left(\frac{\pi}{4},\ 7\right)$ and the local maximum at B is $\left(\frac{5\pi}{4},\ 7\right)$.

b **i** $h'(x)$

$$= \frac{(2b\cos 2x)(c-\sin 2x) - (a + b\sin 2x)(-2\cos 2x)}{(c-\sin 2x)^2}$$

$$= \frac{2bc\cos 2x - \cancel{2b\sin 2x\cos 2x} + 2a\cos 2x + \cancel{2b\sin 2x\cos 2x}}{(c-\sin 2x)^2}$$

$$= \frac{2\cos 2x(a+bc)}{(c-\sin 2x)^2}$$

ii $h'(x) = 0 \ \Leftrightarrow\ x = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$ {as in **b**}

$\therefore\ M = h\left(\frac{\pi}{4}\right) = \dfrac{a + b(1)}{c-1} = \dfrac{a+b}{c-1}$

and $m = h\left(\frac{3\pi}{4}\right) = \dfrac{a + b(-1)}{c-(-1)} = \dfrac{a-b}{c+1}$

iii $H = M - m = \dfrac{a+b}{c-1} - \dfrac{a-b}{c+1}$

$$= \frac{(a+b)(c+1) - (a-b)(c-1)}{(c-1)(c+1)}$$

$$= \frac{\cancel{ac} + a + bc + \cancel{b} - \cancel{ac} + a + bc - \cancel{b}}{c^2 - 1}$$

$$= \frac{2a + 2bc}{c^2-1}$$

$$= \frac{2(a+bc)}{c^2-1}$$

iv When $a = 12,\ b = 2,\ c = 3,$

$H = \dfrac{2(12+6)}{9-1} = \dfrac{2\times 18}{8} = \dfrac{9}{2}$

Now $f\left(\frac{3\pi}{4}\right) = \dfrac{12 + 2\sin\left(\frac{3\pi}{2}\right)}{3 - \sin\left(\frac{3\pi}{2}\right)}$

$$= \frac{12 + 2(-1)}{3-(-1)} = \frac{10}{4} = \frac{5}{2}$$

and $7 - \frac{5}{2} = \frac{9}{2}$, so the result is verified.

11 **a** L_1: $\mathbf{r} = \begin{pmatrix} -1+2\lambda \\ -\lambda \\ 1+2\lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

L_2: $\mathbf{s} = \begin{pmatrix} 1+3\mu \\ 1-\mu \\ 2+2\mu \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

b The lines intersect if $\mathbf{r} = \mathbf{s}$

$\therefore\ -1 + 2\lambda = 1 + 3\mu$ .... (1)

$-\lambda = 1 - \mu$ .... (2)

$1 + 2\lambda = 2 + 2\mu$ .... (3)

From (2), $\lambda = \mu - 1$

So, in (1) $-1 + 2\mu - 2 = 1 + 3\mu$

$\therefore\ \mu = -4$ and $\lambda = -5$

However, in (3) $1 + 2(-5) = -9$ and $2 + 2(-4) = -6$

$\therefore$ (3) is not satisfied by the solutions from (1) and (2).

$\therefore$ L_1 and L_2 do not intersect.

Also, $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$\therefore$ L_1 and L_2 are not parallel.

Thus L_1 and L_2 are skew lines.

c A vector perpendicular to L_1 **and** L_2 is

$$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 3 & -1 & 2 \end{vmatrix}$$

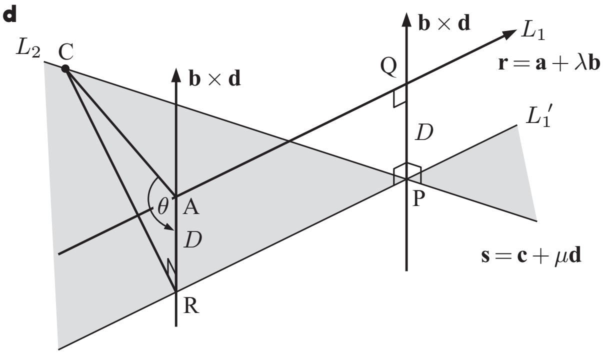
$$= \begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix}\mathbf{k}$$

$$= (-2+2)\mathbf{i} - (4-6)\mathbf{j} + (-2+3)\mathbf{k}$$

$$= 0\mathbf{i} + 2\mathbf{j} + 1\mathbf{k}$$

$$= 2\mathbf{j} + \mathbf{k}$$

d



L_1 is translated to L_1' meeting L_2 at P.

i $\mathbf{b} \times \mathbf{d}$ is perpendicular to L_1 and L_2, so $\mathbf{b} \times \mathbf{d}$ is a normal to the shaded plane containing L_1 and L_2.

ii $\text{A}\widehat{\text{R}}\text{C} = 90°$ {AR is normal to the shaded plane}

Let $\text{C}\widehat{\text{A}}\text{R} = \theta$ be the angle between $\overrightarrow{\text{AC}}$ and $\mathbf{b} \times \mathbf{d}$.

$\therefore\ \cos\theta = \dfrac{\text{AR}}{\text{AC}} = \dfrac{D}{|\overrightarrow{\text{AC}}|}$

$\therefore\ D = |\overrightarrow{\text{AC}}|\cos\theta$

$$= \frac{|\mathbf{c}-\mathbf{a}|\,|\mathbf{b}\times\mathbf{d}|\cos\theta}{|\mathbf{b}\times\mathbf{d}|}$$

$$= \frac{|(\mathbf{c}-\mathbf{a}) \bullet (\mathbf{b}\times\mathbf{d})|}{|\mathbf{b}\times\mathbf{d}|}$$

e For the original lines L_1 and L_2,

$$\mathbf{a} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\mathbf{c} - \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{b} \times \mathbf{d} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \{\text{from } \mathbf{c}\}$$

$$\therefore \quad D = \frac{\left| \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right|}{\sqrt{0+4+1}} = \tfrac{3}{\sqrt{5}} \text{ units}$$

12 **a** $v_x = v_0 \cos\theta$

$\therefore \quad x = \int v_0 \cos\theta \, dt$

$\therefore \quad x = (v_0 \cos\theta)t + c$

But when $t = 0, \; x = 0$

$\therefore \quad c = 0$

$\therefore \quad x = (v_0 \cos\theta)t \quad .... \; (1)$

Likewise, $v_y = v_0 \sin\theta - gt$

$\therefore \quad y = \int (v_0 \sin\theta - gt) \, dt$

$\therefore \quad y = (v_0 \sin\theta)t - g\dfrac{t^2}{2} + d$

When $t = 0, \; y = 0$

$\therefore \quad d = 0$

$\therefore \quad y = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \quad .... \; (2)$

b From (1), $t = \dfrac{x}{v_0 \cos\theta}$

$$\therefore \quad y = v_0 \sin\theta \left(\frac{x}{v_0 \cos\theta} \right) - \tfrac{1}{2}g\left(\frac{x}{v_0 \cos\theta} \right)^2$$

$$\therefore \quad y = (\tan\theta)x - \tfrac{1}{2}g \frac{x^2}{{v_0}^2 \cos^2\theta}$$

$$\therefore \quad y = (\tan\theta)x - (\sec^2\theta) \frac{gx^2}{2{v_0}^2}$$

c The path has equation $y = ax + bx^2$ which is a quadratic in x.

$\therefore$ the path is parabolic.

Furthermore, since $b < 0$, the path is concave.

d $\dfrac{dy}{dx} = \tan\theta - \sec^2\theta \, \dfrac{2gx}{2{v_0}^2}$

$$\therefore \quad \frac{dy}{dx} = 0 \quad \Leftrightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{gx}{{v_0}^2 \cos^2\theta}$$

$$\Leftrightarrow \quad x = \frac{{v_0}^2 \sin\theta \cos\theta}{g}$$

$$\Leftrightarrow \quad x = \frac{{v_0}^2 2\sin\theta \cos\theta}{2g}$$

$$\Leftrightarrow \quad x = \frac{{v_0}^2 \sin 2\theta}{2g}$$

The maximum height is

$$= \left(\frac{\sin\theta}{\cancel{\cos\theta}} \right) \left(\frac{{v_0}^2 \sin\theta \cancel{\cos\theta}}{g} \right) - \tfrac{1}{2}g \, \frac{{v_0}^4 \sin^2\theta \cancel{\cos^2\theta}}{g^2 {v_0}^2 \cancel{\cos^2\theta}}$$

$$= \frac{{v_0}^2 \sin^2\theta}{g} - \tfrac{1}{2} \frac{{v_0}^2 \sin^2\theta}{g}$$

$$= \frac{{v_0}^2 \sin^2\theta}{2g}$$

e Range $= 2\left(\dfrac{{v_0}^2 \sin 2\theta}{2g} \right) = \dfrac{{v_0}^2 \sin 2\theta}{g}$ metres.

f The range is maximised when $\sin 2\theta = 1$

$\therefore \quad 2\theta = 90°$

$\therefore \quad \theta = 45°$

g **i** The maximum height $= \dfrac{300^2 \left(\frac{1}{\sqrt{2}} \right)^2}{2 \times 9.81}$ m

≈ 2294 m

ii Range $= \dfrac{300^2 \times 1}{9.81}$

≈ 9174 m

h Using **e**, $\quad 9500 = \dfrac{400^2 \sin 2\theta}{g}$

$\therefore \quad \sin 2\theta \approx 0.582\,47$

$\therefore \quad 2\theta \approx 35.62°$

$\therefore \quad \theta \approx 17.8°$

The angle is about $17.8°$.