

55 $P(n) = 1000 + ae^{kn}$
 Now $P(0) = 2000$
 $\therefore 1000 + a = 2000$ and so $a = 1000$
 Also, $P(12) = 4000$
 $\therefore 1000 + 1000e^{12k} = 4000$
 $\therefore 1000e^{12k} = 3000$
 $\therefore e^{12k} = 3$
 $\therefore e^k = 3^{\frac{1}{12}}$

If $P(n) = 10000$, $1000 + ae^{kn} = 10000$
 $\therefore 1000e^{kn} = 9000$
 $\therefore e^{kn} = 9$
 $\therefore 3^{\frac{n}{12}} = 3^2$
 $\therefore n = 24$ months

It will take 2 years for the population to reach 10000.

56 a Since a is a solution of the equation,
 $3a^3 - 11a^2 + 8a = 12a$
 $\therefore 3a^3 - 11a^2 - 4a = 0$
 $\therefore a(3a^2 - 11a - 4) = 0$
 $\therefore a(3a + 1)(a - 4) = 0$
 $\therefore a = 0, -\frac{1}{3}, \text{ or } 4$

b If $a = 0$, $3x^3 - 11x^2 + 8x = 0$
 $\therefore x(3x^2 - 11x + 8) = 0$
 $\therefore x(3x - 8)(x - 1) = 0$
 $\therefore x = 0, \frac{8}{3}, \text{ or } 1$

If $a = -\frac{1}{3}$, $3x^3 - 11x^2 + 8x = 12(-\frac{1}{3})$
 $\therefore 3x^3 - 11x^2 + 8x + 4 = 0$
 $x = a$ is a solution, and so $(3x + 1)$ must be a factor.
 $\therefore 3x^3 - 11x^2 + 8x + 4 = (3x + 1)(x^2 + ax + 4)$
 for some a

Equating coefficients of x^2 gives $-11 = 1 + 3a$
 $\therefore a = -4$

$\therefore (3x + 1)(x^2 - 4x + 4) = 0$
 $\therefore (3x + 1)(x - 2)^2 = 0$
 $\therefore x = -\frac{1}{3} \text{ or } 2$

If $a = 4$, $3x^3 - 11x^2 + 8x = 12(4)$
 $\therefore 3x^3 - 11x^2 + 8x - 48 = 0$
 $x = a$ is a solution, so $(x - 4)$ is a factor.

$\therefore 3x^3 - 11x^2 + 8x - 48 = (x - 4)(3x^2 + ax + 12)$
 for some a
 Equating coefficients of x^2 gives $-11 = a - 12$
 $\therefore a = 1$

$\therefore (x - 4)(3x^2 + x + 12) = 0$
 $x = \frac{-1 \pm \sqrt{1^2 - 4 \times 3 \times 12}}{2 \times 3}$
 $x = 4 \text{ or } \frac{-1 \pm i\sqrt{143}}{6}$

57 $P(x)$ is a real polynomial, so $3 + 2i$ must also be a zero of $P(x)$.

$(3 + 2i) + (3 - 2i) = 6$ and
 $(3 + 2i)(3 - 2i) = 9 + 4 = 13$

So, $x^2 - 6x + 13$ is a factor of $P(x)$.
 $\therefore 2x^3 + mx^2 - (m + 1)x + (3 - 4m)$
 $= (x^2 - 6x + 13)(2x + b)$ for some b
 $= 2x^3 + (b - 12)x^2 + (26 - 6b)x + 13b$

Equating coefficients of x^2 : $m = b - 12 \dots (1)$

Equating constants: $3 - 4m = 13b \dots (2)$

Substituting (1) into (2): $13b = 3 - 4(b - 12)$
 $\therefore 13b = 3 - 4b + 48$
 $\therefore 17b = 51$
 $\therefore b = 3$
 $\therefore m = 3 - 12 = -9$

$\therefore P(x) = (x^2 - 6x + 13)(2x + 3)$
 \therefore the zeros of $P(x)$ are $3 \pm 2i$ and $-\frac{3}{2}$

58 $a^2 \times a^3 + a^2 - a^4 \times a - 2 = 0$
 $\therefore a^2 - 2 = 0$
 $\therefore a = \pm\sqrt{2}$

$\therefore P(z) = 2z^3 + z^2 - 4z - 2$

Since $P(z)$ is the same whether $a = \pm\sqrt{2}$, both $z = \sqrt{2}$ and $z = -\sqrt{2}$ must be zeros of $P(z)$.

Hence $(z - \sqrt{2})(z + \sqrt{2}) = (z^2 - 2)$ is a factor of $P(z)$.

$\therefore P(z) = (z^2 - 2)(2z + 1)$
 \therefore the zeros of $P(x)$ are $\pm\sqrt{2}$ and $-\frac{1}{2}$

59 $(f \circ g)(x) = f(g(x))$
 $= f(2x^3)$
 $= 2(2x^3) - 1$
 $= 4x^3 - 1$

So the function $(f \circ g)^{-1}$ is $x = 4y^3 - 1$
 $\therefore 4y^3 = x + 1$
 $\therefore y^3 = \frac{x + 1}{4}$
 $\therefore y = \left(\frac{x + 1}{4}\right)^{\frac{1}{3}}$

So, $(f \circ g)^{-1} : x \mapsto \left(\frac{x + 1}{4}\right)^{\frac{1}{3}}$.

60 Let $P(x) = x^4 + 2x^3 + 8x^2 + 6x + 15$

Since bi is a zero of the real polynomial $P(x)$, so is $-bi$.

$\therefore x^2 + b^2$ is a factor of $P(x)$
 $\therefore P(x) = x^4 + 2x^3 + 8x^2 + 6x + 15$
 $= (x^2 + b^2)(x^2 + cx + d)$ for some c, d
 $= x^4 + cx^3 + (b^2 + d)x^2 + b^2cx + b^2d$

Equating the coefficients of x^3 : $c = 2$

Equating the coefficients of x : $2b^2 = 6$
 $\therefore b = \pm\sqrt{3}$

Equating the coefficients of x^2 : $3 + d = 8$
 $\therefore d = 5$

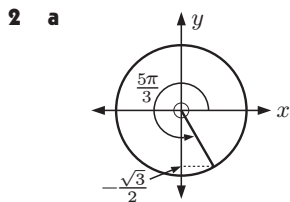
$\therefore P(x) = (x^2 + 3)(x^2 + 2x + 5)$

Now $x^2 + 2x + 5 = 0$
 when $x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1}$
 $= -1 \pm 2i$

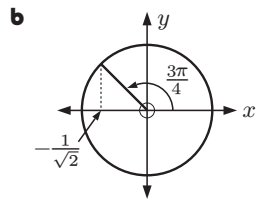
\therefore the zeros are $\pm\sqrt{3}i, -1 \pm 2i$

SOLUTIONS TO TOPIC 3 (CIRCULAR FUNCTIONS AND TRIGONOMETRY)

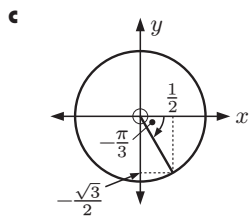
1 a $\frac{2\pi}{9}$ radians **b** 140°
 $= \left(\frac{2\pi}{9} \times \frac{180}{\pi}\right)^\circ$ $= (140 \times \frac{\pi}{180})$ radians
 $= 40^\circ$ $= \frac{7\pi}{9}$



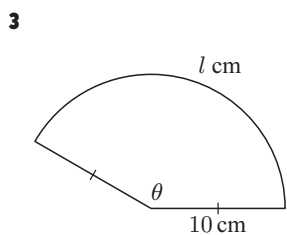
$$\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$



$$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$



$$\begin{aligned} \tan\left(-\frac{\pi}{3}\right) &= \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\ &= -\sqrt{3} \end{aligned}$$

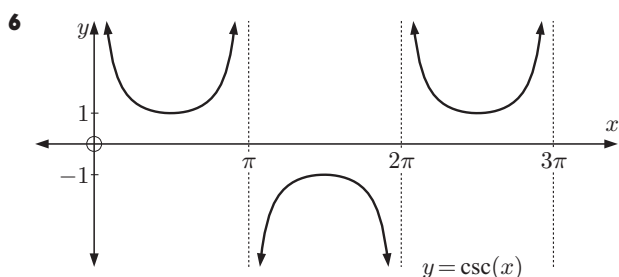


$$\begin{aligned} \text{perimeter} &= 40 \text{ cm} \\ \therefore 10 + 10 + l &= 40 \\ \therefore l &= 20 \\ \text{area} &= \frac{1}{2}\theta r^2 \\ &= \frac{1}{2}lr \quad \{l = \theta r\} \\ &= \frac{1}{2} \times 20 \times 10 \\ &= 100 \text{ cm}^2 \end{aligned}$$

- 4 a** A vertical stretch with factor 2, and a horizontal stretch with factor 3.
b A translation of $\frac{\pi}{3}$ units to the left, and a translation of 4 units downwards.

- 5 a** Amplitude = 1
 The principal axis is $y = 0$
 Period = $\frac{2\pi}{4} = \frac{\pi}{2}$

- b** Amplitude = 2
 The principal axis is $y = -1$
 Period = $\frac{2\pi}{\frac{1}{2}} = 4\pi$



7

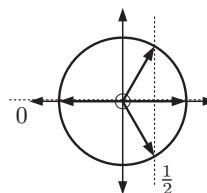
$$\begin{aligned} &\sin\left(\frac{3\pi}{2} - \phi\right) \tan(\phi + \pi) \\ &= \left(\sin\frac{3\pi}{2} \cos\phi - \cos\frac{3\pi}{2} \sin\phi\right) \tan\phi \\ &= ((-1) \cos\phi - 0 \times \sin\phi) \frac{\sin\phi}{\cos\phi} \\ &= -\sin\phi \end{aligned}$$

8

$$\begin{aligned} \cos 2x &= \frac{5}{8} \\ \therefore 1 - 2\sin^2 x &= \frac{5}{8} \quad \{\text{double angle formula}\} \\ \therefore 2\sin^2 x &= \frac{3}{8} \\ \therefore \sin x &= \pm \frac{\sqrt{3}}{4} \end{aligned}$$

9

$$\begin{aligned} \sin 2x &= \sin x, \quad x \in [-\pi, \pi] \\ \therefore 2\sin x \cos x - \sin x &= 0 \\ \therefore \sin x(2\cos x - 1) &= 0 \\ \therefore \sin x = 0 \text{ or } \cos x = \frac{1}{2} \end{aligned}$$



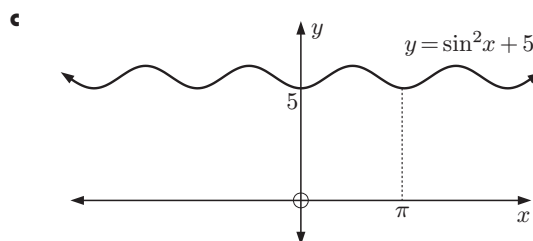
$$\therefore x = 0, \pm\frac{\pi}{3}, \text{ or } \pm\pi$$

10

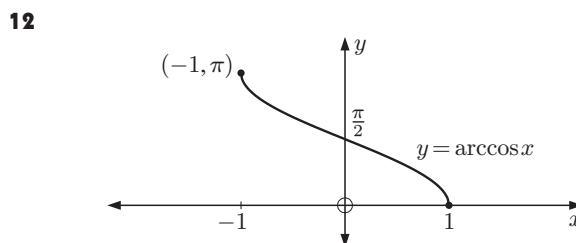
$$\begin{aligned} \text{area} &= 20 \text{ cm}^2 \\ \therefore \frac{1}{2}\theta r^2 &= 20 \\ \therefore \frac{1}{2}lr &= 20 \quad \{l = \theta r\} \\ \therefore \frac{1}{2}(6)r &= 20 \\ \therefore r &= \frac{20}{3} \text{ cm} \end{aligned}$$

So, $\theta = \frac{l}{r} = \frac{6}{\frac{20}{3}} = 0.9$

- 11 a** Period = $\frac{2\pi}{3}$
b Period = $\frac{2\pi}{\frac{1}{2}} = 4\pi$



$$\begin{aligned} \text{Period} &= \pi \\ \text{or } \sin^2 x + 5 &= \left(\frac{1}{2} - \frac{1}{2} \cos 2x\right) + 5 \\ &= -\frac{1}{2} \cos 2x + \frac{11}{2} \\ \therefore \text{period} &= \frac{2\pi}{2} = \pi \end{aligned}$$



13

$$\begin{aligned} 1 - \frac{\sin^2 \theta}{1 + \cos \theta} &= \frac{1 + \cos \theta}{1 + \cos \theta} - \frac{\sin^2 \theta}{1 + \cos \theta} \\ &= \frac{1 + \cos \theta - (1 - \cos^2 \theta)}{1 + \cos \theta} \\ &= \frac{\cos \theta + \cos^2 \theta}{1 + \cos \theta} \\ &= \frac{\cos \theta(1 + \cos \theta)}{1 + \cos \theta} \\ &= \cos \theta \end{aligned}$$

14

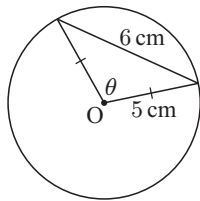
$$\begin{aligned} \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} & \tan 3\theta &= \tan(2\theta + \theta) \\ &= \frac{2 \times 2}{1 - 2^2} & &= \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta} \\ &= -\frac{4}{3} & &= \frac{-\frac{4}{3} + 2}{1 - \left(-\frac{4}{3}\right) \times 2} \\ & & &= \frac{2}{11} \end{aligned}$$

$$\begin{aligned}
 15 \quad \csc(2x) - \cot(2x) &= \frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x} \\
 &= \frac{1 - \cos 2x}{\sin 2x} \\
 &= \frac{1 - (1 - 2\sin^2 x)}{2\sin x \cos x} \\
 &\quad \{\text{double angle formulae}\} \\
 &= \frac{2\sin^2 x}{2\sin x \cos x} \\
 &= \frac{\sin x}{\cos x} \\
 &= \tan x
 \end{aligned}$$

$$\begin{aligned}
 \therefore \tan\left(\frac{5\pi}{12}\right) &= \csc\left(\frac{5\pi}{6}\right) - \cot\left(\frac{5\pi}{6}\right) \\
 &= \frac{1}{\sin\left(\frac{5\pi}{6}\right)} - \frac{1}{\tan\left(\frac{5\pi}{6}\right)} \\
 &= \frac{1}{\frac{1}{2}} - \frac{1}{-\frac{1}{\sqrt{3}}} \\
 &= 2 + \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 16 \quad \cos 2\alpha &= \sin^2 \alpha \\
 \therefore 1 - 2\sin^2 \alpha &= \sin^2 \alpha \\
 \therefore 1 &= 3\sin^2 \alpha \\
 \therefore \sin^2 \alpha &= \frac{1}{3} \\
 \therefore \cos^2 \alpha &= \frac{2}{3} \\
 \therefore \cot^2 \alpha &= \frac{\cos^2 \alpha}{\sin^2 \alpha} = 2 \\
 \therefore \cot \alpha &= \pm\sqrt{2}
 \end{aligned}$$

17



$$\begin{aligned}
 \cos \theta &= \frac{5^2 + 5^2 - 6^2}{2 \times 5 \times 5} \\
 &\quad \{\text{cosine rule}\} \\
 \therefore \cos \theta &= \frac{14}{50} \\
 \therefore \theta &\approx 1.287
 \end{aligned}$$

$$\begin{aligned}
 \text{area of triangle} &= \frac{1}{2} \times 5 \times 5 \times \sin \theta \\
 &= 12 \text{ cm}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{area of sector} &= \frac{1}{2} \times \theta \times 5^2 \\
 &\approx 16.088 \text{ cm}^2
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{area of minor segment} &= \text{area of sector} - \text{area of triangle} \\
 &\approx 4.09 \text{ cm}^2
 \end{aligned}$$

18 For the sine function $y = a \sin b(x - c) + d$:

The amplitude = 2, so $a = 2$.

The period = π , so $\frac{2\pi}{b} = \pi \Rightarrow b = 2$.

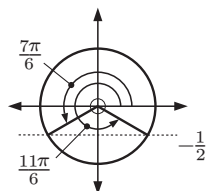
The principal axis is $y = 1$, so $d = 1$.

There is no horizontal translation, so $c = 0$.

\therefore the function is $y = 2 \sin(2x) + 1$

We want to solve $2 \sin(2x) + 1 = 0$, $0 \leq x \leq \pi$

$$\therefore \sin 2x = -\frac{1}{2}$$



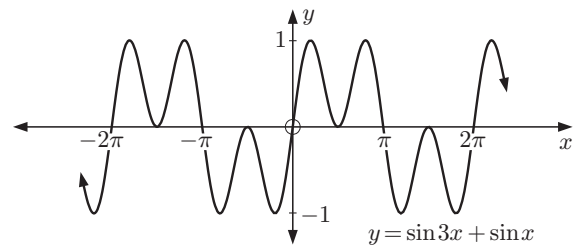
$$\begin{aligned}
 \therefore 2x &= \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \\
 \therefore x &= \frac{7\pi}{12} \text{ or } \frac{11\pi}{12}
 \end{aligned}$$

So, P is $\left(\frac{7\pi}{12}, 0\right)$ and
Q is $\left(\frac{11\pi}{12}, 0\right)$.

19 a Period = $\frac{2\pi}{\frac{1}{3}} = 6\pi$

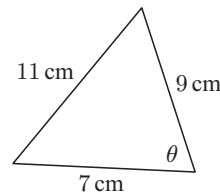
b Period = $\frac{\pi}{5}$

c



$y = \sin 3x$ has period $\frac{2\pi}{3}$, and $y = \sin x$ has period 2π .
So, $y = \sin 3x + \sin x$ has period 2π .

20



The largest angle is opposite the longest side.

$$\cos \theta = \frac{9^2 + 7^2 - 11^2}{2 \times 7 \times 9}$$

{cosine rule}

$$\begin{aligned}
 \therefore \cos \theta &= \frac{9}{126} \\
 \therefore \theta &\approx 85.9^\circ
 \end{aligned}$$

21 a $\csc(x) = \frac{1}{\sin x}$

\therefore vertical asymptotes occur when $\sin x = 0$

\therefore the vertical asymptotes are $x = 0, \pm\pi$, and $\pm 2\pi$

b $\sec(2x) = \frac{1}{\cos 2x}$

\therefore vertical asymptotes occur when $\cos 2x = 0$

$$\therefore 2x = \pm\frac{\pi}{2} + k2\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \pm\frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

\therefore the vertical asymptotes are

$$x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \text{ and } \pm\frac{7\pi}{4}.$$

c $\cot\left(\frac{x}{2}\right) = \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$

\therefore vertical asymptotes occur when $\sin\left(\frac{x}{2}\right) = 0$

$$\therefore \frac{x}{2} = 0 + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = 2k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = 0 \text{ and } x = \pm 2\pi$$

\therefore the vertical asymptotes are $x = 0$ and $x = \pm 2\pi$.

$$\begin{aligned}
 22 \quad \cos 79^\circ \cos 71^\circ - \sin 79^\circ \sin 71^\circ &= \cos(79^\circ + 71^\circ) \\
 &= \cos(150^\circ) \\
 &= -\frac{\sqrt{3}}{2}
 \end{aligned}$$

23

$$\tan 2A = \sin A$$

$$\therefore \frac{\sin 2A}{\cos 2A} = \sin A$$

$$\therefore \frac{2 \sin A \cos A}{2 \cos^2 A - 1} = \sin A \quad \{\text{double angle formula}\}$$

$$\therefore \frac{2 \cos A}{2 \cos^2 A - 1} = 1 \quad \{\sin A \neq 0\}$$

$$2 \cos A = 2 \cos^2 A - 1$$

$$\therefore 2 \cos^2 A - 2 \cos A - 1 = 0$$

$$\therefore \cos A = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 2 \times (-1)}}{2 \times 2}$$

$$= \frac{1 \pm \sqrt{3}}{2}$$

$$\text{But } |\cos A| \leq 1, \text{ so } \cos A = \frac{1 - \sqrt{3}}{2}$$

$$\begin{aligned}
 24 \quad \sin x - 2 \cos x &= A \sin(x + \alpha) \\
 &= A(\sin x \cos \alpha + \cos x \sin \alpha) \\
 &= A \sin x \cos \alpha + A \cos x \sin \alpha
 \end{aligned}$$

Equating the coefficients of $\sin x$ and $\cos x$:

$$\begin{aligned}
 A \cos \alpha &= 1 \quad \text{and} \quad A \sin \alpha = -2 \\
 \therefore \cos \alpha &= \frac{1}{A} \quad \text{and} \quad \sin \alpha = \frac{-2}{A}
 \end{aligned}$$

$$\text{Now } \sin^2 \alpha + \cos^2 \alpha = 1$$

$$\therefore \left(\frac{-2}{A}\right)^2 + \left(\frac{1}{A}\right)^2 = 1$$

$$\therefore \frac{4+1}{A^2} = 1$$

$$\therefore A^2 = 5$$

$$\therefore A = \sqrt{5} \quad \{A > 0\}$$

$$\text{So } \cos \alpha = \frac{1}{\sqrt{5}}, \quad \sin \alpha = -\frac{2}{\sqrt{5}}$$

$\therefore \alpha$ is in the 4th quadrant.

$$\therefore \alpha \approx 5.18$$

$$25 \quad 2 \sin^2 x - \cos x = 1, \quad x \in [0, 2\pi]$$

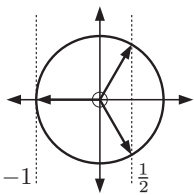
$$\therefore 2(1 - \cos^2 x) - \cos x = 1$$

$$\therefore 2 - 2 \cos^2 x - \cos x = 1$$

$$\therefore 2 \cos^2 x + \cos x - 1 = 0$$

$$\therefore (2 \cos x - 1)(\cos x + 1) = 0$$

$$\therefore \cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -1$$



$$\therefore x = \frac{\pi}{3}, \pi, \text{ or } \frac{5\pi}{3}$$

$$26 \quad \arcsin(2x - 3) = -\frac{\pi}{6}$$

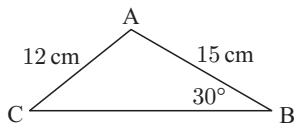
$$\therefore 2x - 3 = \sin\left(-\frac{\pi}{6}\right)$$

$$\therefore 2x - 3 = -\frac{1}{2}$$

$$\therefore 2x = \frac{5}{2}$$

$$\therefore x = \frac{5}{4}$$

27

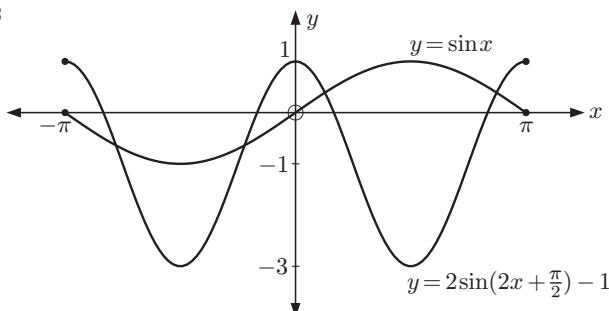


$$\frac{\sin C}{15} = \frac{\sin 30^\circ}{12} \quad \{\text{sine rule}\}$$

$$\therefore C = \arcsin\left(\frac{15 \sin 30^\circ}{12}\right)$$

$$\therefore C \approx 38.7^\circ \quad \text{or} \quad 141.3^\circ$$

28



$$29 \quad \arcsin\left(-\frac{1}{2}\right) + \arctan(1) + \arccos\left(-\frac{1}{2}\right)$$

$$= -\frac{\pi}{6} + \frac{\pi}{4} + \frac{2\pi}{3}$$

$$= \frac{3\pi}{4}$$

$$30 \quad \sin^2 \theta + \cos^2 \theta = 1$$

$$\therefore \left(\frac{2}{3}\right)^2 + \cos^2 \theta = 1$$

$$\therefore \cos^2 \theta = \frac{5}{9}$$

$$\therefore \cos \theta = -\frac{\sqrt{5}}{3} \quad \{\theta \text{ is obtuse}\}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$= 2 \times \frac{2}{3} \times -\frac{\sqrt{5}}{3}$$

$$= -\frac{4\sqrt{5}}{9}$$

31

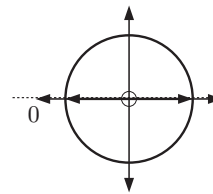
$$\sin x + \cos x = 1, \quad 0 \leq x \leq \pi$$

$$\therefore \sin^2 x + 2 \sin x \cos x + \cos^2 x = 1 \quad \dots (*)$$

{squaring both sides}

$$\therefore \sin 2x + 1 = 1$$

$$\therefore \sin 2x = 0$$



$$\therefore 2x = 0 + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{k\pi}{2}, \quad k \in \mathbb{Z}$$

$$\therefore x = 0, \frac{\pi}{2}, \pi \quad \{0 \leq x \leq \pi\}$$

Since we squared both sides at (*), we need to check our solutions.

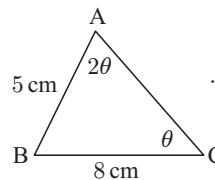
$$\sin 0 + \cos 0 = 1 \quad \checkmark$$

$$\sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1 \quad \checkmark$$

$$\sin \pi + \cos \pi = -1 \quad \times$$

$$\text{So } x = 0 \quad \text{or} \quad \frac{\pi}{2}$$

32 a



$$\frac{\sin 2\theta}{8} = \frac{\sin \theta}{5} \quad \{\text{sine rule}\}$$

$$\therefore \frac{2 \sin \theta \cos \theta}{8} = \frac{\sin \theta}{5}$$

$$\therefore \cos \theta = \frac{4}{5}$$

b

$$\widehat{ABC} = \pi - 3\theta$$

$$= \pi - 3 \arccos\left(\frac{4}{5}\right)$$

$$\approx 1.211$$

$$\text{Area of triangle} \approx \frac{1}{2} \times 5 \times 8 \times \sin(1.211)$$

$$\approx 18.7 \text{ cm}^2$$

$$33 \quad \tan 2x \text{ has period } \frac{\pi}{2}, \text{ and } \tan 3x \text{ has period } \frac{\pi}{3}.$$

The lowest common multiple of $\frac{\pi}{2}$ and $\frac{\pi}{3}$ is π .

$$\therefore \text{the period} = \pi$$

34

$$\cot \theta + \tan \theta = 2, \quad \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

$$\therefore \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} = 2$$

$$\therefore \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} = 2$$

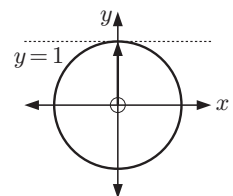
$$\therefore 1 = 2 \sin \theta \cos \theta$$

$$\therefore 1 = \sin 2\theta$$

$$\therefore 2\theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$$

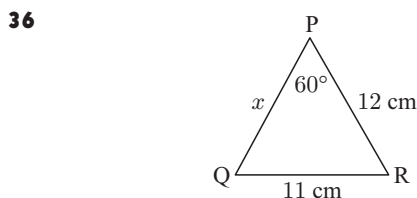
$$\therefore \theta = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \theta = \frac{\pi}{4} \quad \{\text{since } \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\}$$



35 $\tan 2\theta = 2$
 $\therefore \frac{2 \tan \theta}{1 - \tan^2 \theta} = 2$ {double angle formula}
 $2 \tan \theta = 2 - 2 \tan^2 \theta$
 $\tan^2 \theta + \tan \theta - 1 = 0$
 $\therefore \tan \theta = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-1)}}{2 \times 1}$
 $= \frac{-1 \pm \sqrt{5}}{2}$

Now $2\theta \in [\pi, \frac{3\pi}{2}]$, so $\theta \in [\frac{\pi}{2}, \frac{3\pi}{4}]$
 $\therefore \tan \theta = \frac{-1 - \sqrt{5}}{2}$ { $\tan \theta < 0$ }



$11^2 = x^2 + 12^2 - 2 \times x \times 12 \cos 60^\circ$
 $\therefore 121 = x^2 + 144 - 12x$
 $\therefore x^2 - 12x + 23 = 0$
 $\therefore x = \frac{12 \pm \sqrt{(-12)^2 - 4 \times 1 \times 23}}{2 \times 1}$
 $\therefore x = 6 \pm \sqrt{13}$, so $PQ = 6 \pm \sqrt{13}$ cm

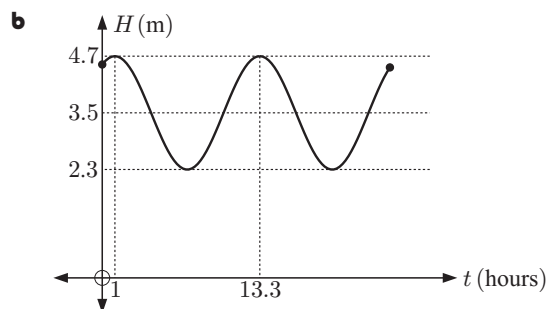
37 $\frac{1}{\tan \theta - \sec \theta} = \frac{1}{\frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta}}$ {provided $\cos \theta \neq 0$ }
 $= \frac{1}{\frac{\sin \theta - 1}{\cos \theta}}$
 $= \frac{\cos \theta}{\sin \theta - 1} \times \frac{(\sin \theta + 1)}{(\sin \theta + 1)}$
 $= \frac{\cos \theta \sin \theta + \cos \theta}{\sin^2 \theta - 1}$
 $= \frac{\cos \theta \sin \theta + \cos \theta}{-\cos^2 \theta}$
 $= -\frac{\cos \theta \sin \theta}{\cos^2 \theta} - \frac{\cos \theta}{\cos^2 \theta}$
 $= -\frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta}$
 $= -(\tan \theta + \sec \theta)$

38 a $\sqrt{3} \tan \left(\frac{x}{2}\right) = -1$, $x \in [-\pi, 3\pi]$
 $\therefore \tan \left(\frac{x}{2}\right) = -\frac{1}{\sqrt{3}}$
 $\therefore \frac{x}{2} = \frac{5\pi}{6} + k\pi$, $k \in \mathbb{Z}$
 $\therefore x = \frac{5\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$
 $\therefore x = \frac{5\pi}{3}$ or $-\frac{\pi}{3}$ { $x \in [-\pi, 3\pi]$ }

b $\sqrt{3} + 2 \sin(2x) = 0$, $x \in [-\pi, 3\pi]$
 $\therefore \sin 2x = -\frac{\sqrt{3}}{2}$
 $\therefore 2x = \left. \begin{matrix} \frac{4\pi}{3} \\ \frac{5\pi}{3} \end{matrix} \right\} + k2\pi$, $k \in \mathbb{Z}$
 $\therefore x = \left. \begin{matrix} \frac{2\pi}{3} \\ \frac{5\pi}{6} \end{matrix} \right\} + k\pi$, $k \in \mathbb{Z}$
 $\therefore x = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}, \frac{8\pi}{3}$, or $\frac{17\pi}{6}$
{ $x \in [-\pi, 3\pi]$ }

39 $\sin x = 2 \sin \left(x - \frac{\pi}{6}\right)$
 $\therefore \sin x = 2 \left(\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}\right)$
 $= 2 \sin x \left(\frac{\sqrt{3}}{2}\right) - 2 \cos x \left(\frac{1}{2}\right)$
 $\therefore \sin x(1 - \sqrt{3}) = -\cos x$
 $\therefore \frac{\sin x}{\cos x} = -\frac{1}{1 - \sqrt{3}}$
 $\therefore \tan x = \frac{1}{\sqrt{3} - 1}$

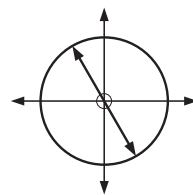
40 a Consider the sine function model $H(t) = a \sin b(t-c) + d$
The amplitude $a = \frac{2.4}{2} = 1.2$ m.
The period is 12.3 hours, so $\frac{2\pi}{b} = 12.3$
 $\therefore b \approx 0.5108$
The principal axis is $H = 4.7 - 1.2 = 3.5$ m so $d = 3.5$.
The first low tide is at $t = 1 + 6.15 = 7.15$, and the next high tide is at $t = 1 + 12.3 = 13.3$
 $\therefore c = \frac{7.15 + 13.3}{2} \approx 10.2$
 \therefore the model is $H(t) \approx 1.2 \sin(0.5108(t - 10.2)) + 3.5$
where t is the time in hours after midnight,
 $0 \leq t \leq 24$.



41 a $\sin \left(\arccos \left(-\frac{\sqrt{3}}{2}\right)\right)$
 $= \sin \left(\frac{5\pi}{6}\right)$
 $= \frac{1}{2}$

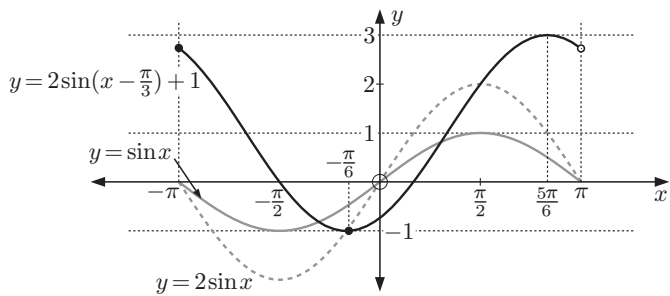
b $\tan \left(\arcsin \frac{1}{\sqrt{2}}\right)$
 $= \tan \left(\frac{\pi}{4}\right)$
 $= 1$

42 $\sin x + \sqrt{3} \cos x = 0$
 $\therefore \sin x = -\sqrt{3} \cos x$
 $\therefore \frac{\sin x}{\cos x} = -\sqrt{3}$
 $\therefore \tan x = -\sqrt{3}$
 $\therefore x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$ { $x \in [0, 2\pi]$ }



43 $\frac{\sin \theta + 2 \cos \theta}{\sin \theta - \cos \theta} = 2$
 $\therefore \sin \theta + 2 \cos \theta = 2(\sin \theta - \cos \theta)$
 $\therefore 4 \cos \theta = \sin \theta$
 $\therefore \tan \theta = 4$
 $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
 $= \frac{2 \times 4}{1 - 4^2}$
 $= -\frac{8}{15}$

44 $y = 2 \sin \left(x - \frac{\pi}{3}\right) + 1$ is a translation of $y = 2 \sin x$ by $\left(\frac{\pi}{3}, 1\right)$.
So, we start with $y = \sin x$, we stretch it vertically with scale factor 2 to produce $y = 2 \sin x$, then perform the translation.



- 45 $\cos 2x + \sqrt{3} \sin 2x = 1, x \in [-\pi, \pi]$
 $\therefore 1 - 2\sin^2 x + 2\sqrt{3} \sin x \cos x = 1$
 $\therefore -2\sin^2 x + 2\sqrt{3} \sin x \cos x = 0$
 $\therefore 2\sin x(\sqrt{3} \cos x - \sin x) = 0$
 $\therefore \sin x = 0$ or $\sqrt{3} \cos x = \sin x$
 $\therefore \sin x = 0$ or $\tan x = \sqrt{3}$
 $\therefore x = -\pi, -\frac{2\pi}{3}, 0, \frac{\pi}{3}, \text{ or } \pi$

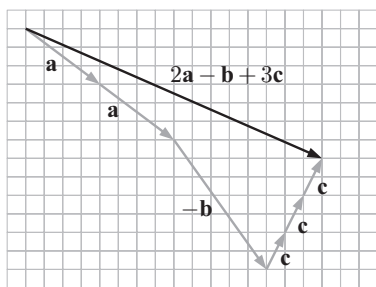
SOLUTIONS TO TOPIC 4 (VECTORS)

1 a $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

b magnitude $= \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$ units

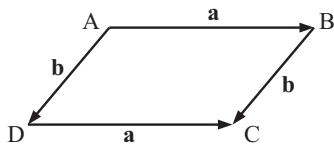
c The unit vector in the opposite direction is $-\frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

2



$$\begin{aligned} 2\mathbf{a} - \mathbf{b} + 3\mathbf{c} &= 2 \begin{pmatrix} 4 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ 7 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 8 + 5 + 3 \\ -6 - 7 + 6 \end{pmatrix} \\ &= \begin{pmatrix} 16 \\ -7 \end{pmatrix} \text{ which checks with the diagram.} \end{aligned}$$

3 a



- b $\vec{BC} = \vec{AD} = \mathbf{b}$ and $\vec{CD} = -\vec{AB} = -\mathbf{a}$
 $\vec{AC} = \vec{AB} + \vec{BC} = \mathbf{a} + \mathbf{b}$ and $\vec{BD} = \vec{BC} + \vec{CD} = \mathbf{b} - \mathbf{a}$
- c $\vec{AC} \cdot \vec{BD} = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a})$
 $= \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a}$
 $= \mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} \quad \{\text{as } \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}\}$
 $= |\mathbf{b}|^2 - |\mathbf{a}|^2$
 and, if $|\mathbf{b}| = |\mathbf{a}|$, $\vec{AC} \cdot \vec{BD} = 0$
- d Since $\vec{AC} \cdot \vec{BD} = 0$, \vec{AC} and \vec{BD} are perpendicular.

- 4 a $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$ where θ is the angle between \mathbf{a} and \mathbf{b} .
 If $\mathbf{a} \cdot \mathbf{b} < 0$, then $\cos \theta < 0$ and so $90^\circ < \theta < 180^\circ$.

b $\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = -6 - 1 + 3 = -4$

$$|\mathbf{a}| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{14} \quad \text{and}$$

$$|\mathbf{b}| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$$

$$\text{and } \cos \theta = \frac{-4}{\sqrt{14}\sqrt{11}} \approx -0.3223 \quad \text{and so } \theta \approx 108.8^\circ.$$

- 5 a $\begin{pmatrix} k \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ k \\ 3k \end{pmatrix}$ are parallel if $\begin{pmatrix} 4 \\ k \\ 3k \end{pmatrix} = a \begin{pmatrix} k \\ 1 \\ 3 \end{pmatrix}$

for some a .

$$\text{Thus, } 4 = ak \quad \dots (1)$$

$$k = a \quad \dots (2)$$

$$3k = 3a \quad \dots (3)$$

From (1) and (2), $k^2 = 4$ and so $k = \pm 2$.

Hence the vectors are parallel if $k = \pm 2$.

- if $k = 2$, the vectors are $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$

- if $k = -2$, the vectors are $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -2 \\ -6 \end{pmatrix}$

- b The vectors are perpendicular if $\begin{pmatrix} k \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ k \\ 3k \end{pmatrix} = 0$

$$\therefore 4k + k + 9k = 0 \quad \text{and so } k = 0.$$

So, the vectors $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ are perpendicular.

- 6 a i The equation can be written as

$$\frac{x-1}{2} = \frac{3-y}{3} = z = t$$

$$\therefore x = 2t + 1, y = -3t + 3, z = t$$

$$\text{or } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}.$$

$$\therefore \text{a vector parallel to the line is } \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}.$$

ii Letting $t = 0$, a point on the line is $(1, 3, 0)$.

iii The point $(7, -3, 2)$ lies on the line if

$$7 = 2t + 1 \quad \dots (1)$$

$$-3 = -3t + 3 \quad \dots (2)$$

$$2 = t \quad \dots (3)$$

So, from (3), $t = 2$ and from (1), $t = 3$ which is not possible. Thus the point $(7, -3, 2)$ does not lie on the line.

- b There are many possible answers.

Since $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, a possible

line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, which is

$$x = 5, y = -3 + s, z = 2 + 3s, s \in \mathbb{R}.$$