12 **a** When $P(x)$ is divided by $(x-a)^2$, the quotient is $Q(x)$ and the remainder is $bx+c$.

i Write $P(x)$ in terms of $Q(x)$, $(x-a)^2$, and $bx+c$. (1 mark)

ii Hence find $P(a)$ and $P'(a)$. (2 marks)

iii Deduce that the remainder is $P'(a)(x-a)+P(a)$. (3 marks)

iv Find the remainder when $P(x)=x^5$ is divided by $(x+2)^2$. (1 mark)

b Using the substitution $x=3\cos\theta$, find $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$. (6 marks)

TRIAL EXAMINATION 3

NO CALCULATOR — 120 marks / 3 hours

SECTION A (58 marks)

1 When a cubic polynomial $p(x)$ is divided by $x(2x-3)$, the remainder is $ax+b$, where a and b are real.

a If the quotient is the same as the remainder, write down an expression for $p(x)$. (2 marks)

b Prove that $(2x-1)$ and $(x-1)$ are both factors of $p(x)$. (1 mark)

c Find in expanded form, an expression for $p(x)$, given that it has y-intercept 7 and passes through point $(2, 39)$. (3 marks)

2 Suppose $\ln\left(\dfrac{a^2}{b}\right)=k$ and $\ln\left(\dfrac{b^2}{a^3}\right)=2$.

a Show that $b=e^{3k+4}$. (4 marks)

b Find r and s given that $a=e^{rk+s}$. (2 marks)

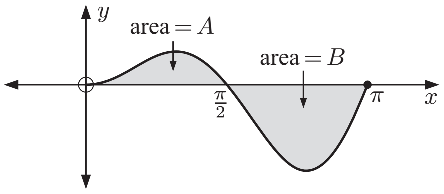
3 **a** Find constants a and b given that $\dfrac{x-2}{x^2-1}=\dfrac{a}{x+1}+\dfrac{b}{x-1}$ for all $x\in\mathbb{R}$. (3 marks)

b Hence evaluate $\displaystyle\int_{-4}^{-2}\frac{x-2}{x^2-1}\,dx$ giving your answer in logarithmic form. (3 marks)

c Explain why $\displaystyle\int_{1}^{3}\frac{x-2}{x^2-1}\,dx$ does not exist. (1 mark)

4 **a** Use integration by parts to show that $\int x\sin 2x\,dx=\frac{1}{4}\sin 2x-\dfrac{x}{2}\cos 2x+c$. (3 marks)

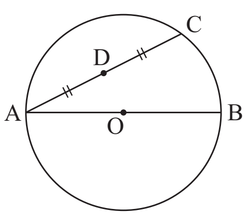
b The graph of $f(x)=x\sin 2x$, $x\in[0,\pi]$ is given below.



Show that $B=3A$. (4 marks)

5 A circle has centre O and diameter [AB]. D is the midpoint of chord [AC].

Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OC}=\mathbf{c}$.



a Find vectors $\overrightarrow{AC}$ and $\overrightarrow{OD}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. (3 marks)

b Hence prove, using vector methods, that [AC] and [OD] are perpendicular. (3 marks)

c What theorem has been proved in **b**? (1 mark)

6 Let $f(x)=x^2+4x$, $x\in\,]-\infty,-2]$ and $g(x)=\sqrt{3-2x}$. Find:

a the domain of $g(x)$ (1 mark)

b $(g\circ f)(-3)$ (2 marks)

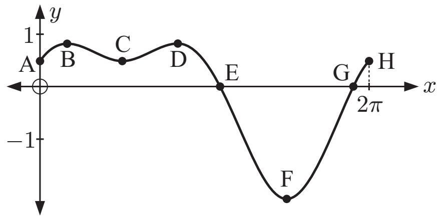
c $f^{-1}(x)$. (3 marks)

7 Solve for x:

a $2^{2x}+2^{x+1}=15$ (3 marks)

b $\sin^2 x+\cos x=1.25$, $x\in[-\pi,\pi]$ (3 marks)

8 The graph of $f(x)=\frac{1}{2}\cos 2x+\sin x$ for $x\in[0,2\pi]$ is illustrated, but not drawn to scale.



a State the coordinates of A and H. (2 marks)

b Find the exact coordinates of the stationary points at B, C, D, and F. (5 marks)

c Find the exact values of the x-intercepts at E and G. (6 marks)

SECTION B (62 marks)

9 **a** Show that:

i $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

ii $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ (3 marks)

b Hence show that:

i $\cos 3\theta+\cos\theta=2\cos 2\theta\cos\theta$

ii $\sin 3\theta+\sin\theta=2\sin 2\theta\cos\theta$ (3 marks)

c If $z=\operatorname{cis}\theta$, what are z^2 and z^3? (1 mark)

d Use **b** to show that $z^3+z=2\cos\theta\operatorname{cis}2\theta$. (3 marks)

e Hence find $\left|z^3+z\right|$ and $\arg(z^3+z)$. (1 mark)

f On an Argand diagram with $\theta\approx 25°$, illustrate z, z^2, z^3, and z^3+z.
Show that $\arg(z^3+z)=\arg(z^2)$. (3 marks)

g Use your Argand diagram to find all values of θ, where $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, such that $z+z^3$ is purely imaginary. (2 marks)

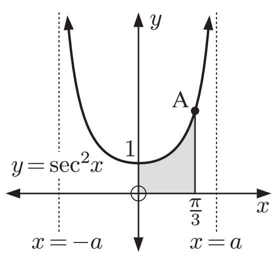
10 **a** **i** If $f(x)=\tan^3 x$, find $f'(x)$ in terms of $\sec x$ only. (2 marks)

ii Hence show that $\int\sec^4 x\,dx=\tan x+\frac{1}{3}\tan^3 x+c$. (3 marks)

b One part of the graph of $y=\sec^2 x$ is shown.
Find:

i the value of a

ii the coordinates of A.



(2 marks)

c Find the exact value of the area of the shaded region. (2 marks)

d If the shaded region is rotated about the x-axis through $360°$, find the volume of the solid generated. (3 marks)

e If the shaded region is rotated about the y-axis through $360°$, explain why the volume of the solid generated is $\frac{4\pi^3}{9} - \pi \int_1^4 \left[\arccos\left(\frac{1}{\sqrt{y}}\right)\right]^2 dy$. (4 marks)

11 Suppose $\mathbf{r} = \begin{pmatrix} 2t+5 \\ -2t-1 \\ t \end{pmatrix}$, $t \in \mathbb{R}$ is an equation of line L.

The plane P has a normal vector $3\mathbf{i} - 4\mathbf{j} - \mathbf{k}$ and passes through the point $A(-1, 0, 4)$.

a Show that the point $B(9, -5, 2)$ lies on the line L. (1 mark)

b Give an equation of the plane P. (2 marks)

c Show that the line L meets the plane P at the point $C(1, 3, -2)$. (2 marks)

d The line N through the point $B(9, -5, 2)$ is perpendicular to the plane P. Find an equation of the line N. (3 marks)

e Show that the point of intersection of the line N and the plane P is the point $D(3, 3, 4)$. (2 marks)

f Find the coordinates of the point B' on the line N such that the plane P bisects the line segment $[BB']$. (2 marks)

g Decide if the vector $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ is parallel to the line (CB'). Give reasons for your answer. (2 marks)

12 **a** Use mathematical induction to prove that $2^{4n+3} + 3^{3n+1}$ is divisible by 11 for all $n \in \mathbb{Z}^+$. (8 marks)

b a, b, and c are consecutive terms of an arithmetic sequence. a, $b+1$, and $c+29$ are consecutive terms of a geometric sequence.
Given that $a + b + c = 33$, find all possible values for a, b, and c. (8 marks)

CALCULATOR — 120 marks / 3 hours

SECTION A — (58 marks)

1 **a** If $z = r \operatorname{cis} \theta$, find in terms of r and θ:

i z^3 **ii** $\sqrt[3]{z}$ (2 marks)

b Given $-11 + ai = (1 - ai)^3$ where $a \in \mathbb{R}$, find the possible values of a. (4 marks)

2 Suppose $f(x) = \begin{cases} \frac{1}{2}e^{-bx}, & 0 \leqslant x \leqslant 1, \quad b \neq 0 \\ 0, & \text{otherwise.} \end{cases}$

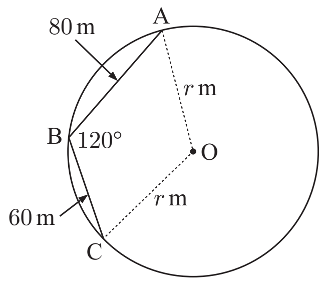
a Show that if $f(x)$ is a well defined PDF, then $e^{-b} + 2b - 1 = 0$. (3 marks)

b Find b correct to 3 decimal places. (1 mark)

c Find the mean μ of the distribution of X. (1 mark)

d Calculate $\text{Var}(X)$. (2 marks)

3 A, B, and C are lamp posts on the edge of a circular lake of radius r metres. O is the lake's centre.



a Use triangle ABC to find the value of $(AC)^2$. (2 marks)

b Use triangle AOC to find the value of $(AC)^2$. (2 marks)

c Calculate the radius of the lake. (1 mark)

d Find the measure of $B\widehat{A}O$. (2 marks)

4 A sample of 20 randomly selected cockles is taken from a population of thousands of cockles, with no replacement. If a cockle's shell is cracked, it is unsaleable. It is known that 3.2% of shells in the population are cracked.

a Explain why this sampling method is not strictly binomial, but the binomial model gives a very good approximation. (2 marks)

b Suppose X = the number of cracked shells in the sample of 20 cockles. Find:

i $P(X \leqslant 2)$ **ii** $P(X \geqslant 4)$. (4 marks)

5 **a** By finding $\frac{d}{dx}(x^{n+1} \ln x)$, show that

$$\int x^n \ln x \, dx = \frac{x^{n+1}}{(n+1)^2}((n+1)\ln x - 1) + c$$

provided $n \neq -1$. (5 marks)

b Find $\int x^n \ln x \, dx$ when $n = -1$. (2 marks)

6 The displacement of a particle moving in a straight line is given by $s = t \sin\left(\frac{t}{2}\right) + 2\cos\left(\frac{t}{2}\right)$ metres, where t is the time in seconds.

a Find an expression for v, the velocity of the particle at time t, in metres per second. (2 marks)

b The particle starts at rest. When does the particle first change direction, and what is its position at that time? (3 marks)

c Find the exact value of the acceleration of the particle after $\frac{\pi}{3}$ seconds. (3 marks)

7 **a** Find the *exact* value of x if

$$\arctan\left(\frac{x}{3}\right) + \arctan 6 = \arctan 3.$$ (5 marks)

b If $y = \arctan\left(\frac{x}{3}\right)$, find $\frac{dy}{dx}$. (3 marks)

8 A circle with equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(0, 3)$, $(2, -1)$, and $(8, 7)$.

a Find three equations involving a, b, and c only. (3 marks)

b Write the equations in augmented matrix form. Simplify the result to echelon form. (3 marks)

c Find the values of a, b, and c. (3 marks)

SECTION B **(62 marks)**

9 The results obtained in a Science examination are distributed approximately normally with mean 56.7 and standard deviation 18.2. All final results are integers.

a A student who did the Science examination is selected at random. Find the probability that the student had a result:

i between 65 and 85 inclusive

ii at least 70. (2 marks)

b An 'A' grade is awarded to a student whose result is in the top 10%.
An 'F' grade is given to a student whose result is in the bottom 15%.
Determine the:

i smallest result for which an 'A' is awarded

ii largest result for which an 'F' is given. (4 marks)

c Four of the students are chosen at random. What is the probability that two will get an 'A' and the other two will get an 'F'? (3 marks)

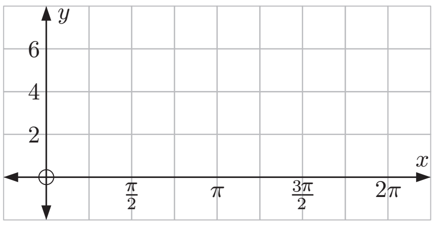
d Suppose 20 of the students are chosen at random.
Find the probability that:

i none receive an 'A'

ii at least three receive an 'A'. (3 marks)

10 **a** Consider the function $f(x) = \dfrac{12 + 2\sin 2x}{3 - \sin 2x}$.

i Sketch the graph of $y = f(x)$ for $x \in [0, 2\pi]$ using the given grid:



(3 marks)

ii Find the exact coordinates of any maxima turning points. (4 marks)

b The formula $h(x) = \dfrac{a + b\sin 2x}{c - \sin 2x}$ gives the profile of plastic sheeting used on the tops of verandahs. The constants a, b, and c are positive.

i Show that $h'(x) = \dfrac{2\cos 2x(a + bc)}{(c - \sin 2x)^2}$. (3 marks)

ii Hence find the maximum (M) and minimum (m) values of $h(x)$. (2 marks)

iii If the height of the profile is H, show that $H = \dfrac{2(a + bc)}{c^2 - 1}$. (2 marks)

iv Check the formula in **iii** using the specific example in part **a**. (2 marks)

11 L_1 and L_2 are two lines in space given by:

$$L_1: \quad \frac{x+1}{2} = -y = \frac{z-1}{2} = \lambda$$

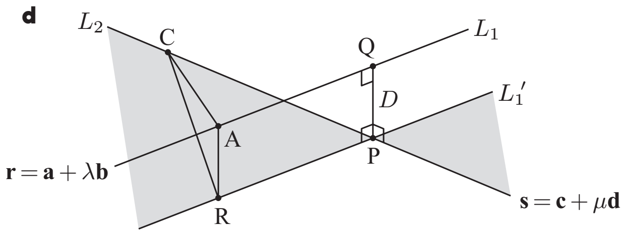
$$L_2: \quad \frac{x-1}{3} = 1 - y = \frac{z-2}{2} = \mu.$$

a Write the equations of L_1 and L_2 in vector form. (2 marks)

b Show that L_1 and L_2 do not intersect, and also that they are not parallel. (5 marks)

c Find a vector which is perpendicular to both L_1 and L_2. (2 marks)

d



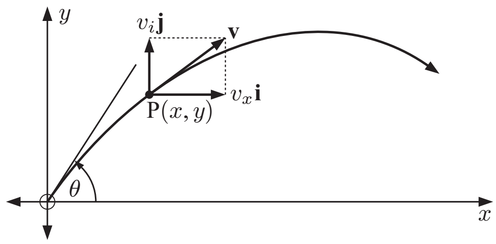
L_1 and L_2 are two skew lines in space. There exist points P on L_2 and Q on L_1, where PQ is the shortest distance between lines L_1 and L_2. When this occurs, [PQ] is perpendicular to both L_1 and L_2.
We now translate L_1 to L_1', where L_1' passes through P, and L_2 and L_1' form the shaded plane.
A and C are two fixed points on L_1 and L_2 respectively.

i Explain why the vector in the direction of [PQ] is $\mathbf{b} \times \mathbf{d}$. (1 mark)

ii In triangle CAR let $\widehat{CAR} = \theta$. Hence show that the shortest distance between L_1 and L_2 is $\dfrac{|(\mathbf{c} - \mathbf{a}) \bullet (\mathbf{b} \times \mathbf{d})|}{|\mathbf{b} \times \mathbf{d}|}$. (4 marks)

e Find the shortest distance between the original lines L_1 and L_2. (3 marks)

12



A missile is launched at an angle θ to the horizontal ground. At the point $P(x, y)$ on its path, the horizontal component of its velocity is $v_x = v_0 \cos\theta$, and the vertical component is $v_y = v_0 \sin\theta - gt$. v_0 is the initial speed of the missile, and g is the gravitational constant.

a Show that at P,
$x = (v_0 \cos\theta)t$ and $y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$. (4 marks)

b By eliminating t, show that $y = (\tan\theta)x - (\sec^2\theta)\dfrac{gx^2}{2{v_0}^2}$. (2 marks)

c What is the nature of the missile's path? Give evidence to support your answer. (1 mark)

d Show that the missile reaches its maximum height when $x = \dfrac{{v_0}^2 \sin 2\theta}{2g}$, and that the maximum height is $\dfrac{{v_0}^2 \sin^2\theta}{2g}$. (4 marks)

e Find the range of the missile, which is the horizontal distance it will travel before landing. (1 mark)

f What value of θ will achieve maximum range for the missile? (1 mark)

g Suppose $v_0 = 300\ \text{m s}^{-1}$, $\theta = 45°$, and $g = 9.81\ \text{m s}^{-2}$.
Find:

i the maximum height reached

ii the range of the missile. (2 marks)

h SPX3 rockets have an initial velocity of 400 m s^{-1}. If the target is 9.5 km away, at what angle must the missile be fired at in order to hit the target? (2 marks)