TOPIC 1: ALGEBRA

SEQUENCES AND SERIES

A **number sequence** is a set of numbers defined by a rule. Often, the rule is a formula for the **general term** or ***n*th term** of the sequence.

A sequence which continues forever is called an **infinite sequence**.

A sequence which terminates is called a **finite sequence**.

Arithmetic Sequences

In an **arithmetic sequence**, each term differs from the previous one by the same fixed number.

$u_{n+1} - u_n = d$ for all n, where d is a constant called the **common difference**.

For an arithmetic sequence with first term u_1 and common difference d, the nth term is $u_n = u_1 + (n-1)\,d$.

Geometric Sequences

In a **geometric sequence**, each term is obtained from the previous one by multiplying by the same non-zero constant.

$\dfrac{u_{n+1}}{u_n} = r$ for all n, where r is a constant called the **common ratio**.

For a geometric sequence with first term u_1 and common ratio r, the nth term is $u_n = u_1 r^{n-1}$.

For **compound interest** problems we have a geometric sequence. If the interest rate is $i\%$ per time period then the common ratio is $\left(1 + \frac{i}{100}\right)$ and the number of compounding periods is n.

Series

A **series** is the addition of the terms of a sequence.

For a finite series with n terms, its sum is $S_n = u_1 + u_2 + ... + u_n$.

For an infinite series, the sum $u_1 + u_2 + ... + u_n + ...$ can only be calculated in some cases.

Using **sigma notation** we write

$u_1 + u_2 + u_3 + ... + u_n$ as $\sum\limits_{k=1}^{n} u_k$.

For a **finite arithmetic series**, $S_n = \frac{n}{2}\,(u_1 + u_n)$ or $S_n = \frac{n}{2}(2u_1 + (n-1)d)$.

For a **finite geometric series** with $r \neq 1$, $S_n = \dfrac{u_1\,(r^n - 1)}{r - 1}$.

The sum of an **infinite geometric series** is

$S = \dfrac{u_1}{1-r}$ provided $|r| < 1$.

If $|r| > 1$ the series is **divergent**.

EXPONENTIALS AND LOGARITHMS

Exponential and logarithmic functions are inverses of each other. The graph of $y = \log_a x$ is the reflection in the line $y = x$ of the graph of $y = a^x$.

Index or Exponent Laws	
$a^m \times a^n = a^{m+n}$	$a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$
$\dfrac{a^m}{a^n} = a^{m-n}$	$a^{\frac{1}{n}} = \sqrt[n]{a}$
$(a^m)^n = a^{mn}$	$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$
$a^0 = 1 \quad (a \neq 0)$	

If $a^x = a^k$ then $x = k$. So, if the base numbers are the same, we can **equate indices**.

If $b = a^x$, $a \neq 1$, $a > 0$, we say that x is the **logarithm** of b in base a, and that $b = a^x \Leftrightarrow x = \log_a b$, $b > 0$.

The **natural logarithm** is the logarithm in base e. $\ln x \equiv \log_e x$

Logarithm Laws	
Base c, $c \neq 1$, $c > 0$	**Base e**
$\log_c AB = \log_c A + \log_c B$	$\ln xy = \ln x + \ln y$
$\log_c \left(\dfrac{A}{B}\right) = \log_c A - \log_c B$	$\ln \left(\dfrac{x}{y}\right) = \ln x - \ln y$
$\log_c A^n = n \log_c A$	$\ln x^y = y \ln x$
$\log_c 1 = 0$	$\ln 1 = 0$
$\log_c c = 1$	$\ln e = 1$

To change the base of a logarithm, use the rule $\log_b a = \dfrac{\log_c a}{\log_c b}$.

$x = \log_a a^x$ and $x = a^{\log_a x}$ provided $x > 0$.

THE BINOMIAL THEOREM

$a + b$ is called a **binomial** as it contains two terms.

Any expression of the form $(a+b)^n$ is called a **power of a binomial**.

The **binomial coefficient** $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

where $n! = n(n-1)(n-2).... \times 3 \times 2 \times 1$
and $0! = 1$

You should also know how to calculate binomial coefficients from Pascal's triangle and using your calculator.

The **general binomial expansion** is

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + ... + \binom{n}{r} a^{n-r}b^r + ... + \binom{n}{n} b^n$$

where $\binom{n}{r}$ is the binomial coefficient of $a^{n-r}b^r$ and $r = 0, 1, 2, 3, ..., n$.

The **general term** in the binomial expansion is

$T_{r+1} = \binom{n}{r} a^{n-r}b^r$, so $(a+b)^n = \sum\limits_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$.

SKILL BUILDER QUESTIONS (NO CALCULATORS)

1 The first four terms of an arithmetic sequence are 51, 45, 39, 33.

a Write down the common difference d.

b Find the 20th term u_{20}.

c Find the sum of the first 20 terms.

2 The first four terms of a geometric sequence are 0.125, 0.25, 0.5, 1.

a Write down the common ratio r.

b Find the 20th term u_{20}.

c Find the sum of the first 10 terms.

3 An infinite geometric series has terms $u_1 = 27$ and $u_4 = 8$.

a Find the common ratio r.

b Find the 6th term of the series.

c Using summation notation, write an expression for the sum to infinity S of the series.

d Evaluate S.

4 An arithmetic series has terms $u_7 = 1$ and $u_{15} = -23$.

a Find the first term u_1 and common difference d.

b Find the 27th term u_{27}.

c Find the sum of the first 27 terms of the series.

5 The first term of a finite arithmetic series is 18 and the sum of the series is -210. The common difference is -3. Suppose there are n terms in the series.

a Show that $\frac{n}{2}(39 - 3n) = -210$.

b Hence find n.

6 Consider the infinite geometric sequence $x^{-\frac{1}{2}}, x, x^{\frac{5}{2}}, ...$

a Write down the common ratio r.

b Find the 10th term of the sequence.

c For what values of x will the sum of the corresponding infinite geometric series converge?

7 A finite geometric series is defined by $\sum_{k=1}^{7} 3 \times 2^{k-1}$.

a How many terms are there in the series?

b Find the first term u_1 and common ratio r.

c Find the sum of the series.

8 **a** An infinite geometric series is defined by $\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^k$.

i Find the first term u_1 and common ratio r.

ii Find the sum of the series.

b A finite arithmetic series is defined by $\sum_{k=1}^{n} (k-4)$.

i Find the first term u_1 and common difference d.

ii Find the sum of the series, in terms of n.

c Find n such that the sums of the series in **a** and **b** are equal.

9 **a** Find the sum to infinity of the infinite geometric series $1 + 0.6 + (0.6)^2 + (0.6)^3 + ...$

b When a ball is dropped from a height of 1 m, on each bounce it returns to 60% of the height it reached previously. Find the total distance travelled by the ball until it stops bouncing.

10 Solve for x:

a $9^x - 6(3^x) + 8 = 0$ **b** $8^{2x-3} = 16^{2-x}$

11 Solve for x:

a $\log_5(2x-1) = -1$ **b** $25^x - 5^{x+1} + 6 = 0$

12 Solve for x:

a $4^x + 4 = 17(2^{x-1})$ **b** $\log_3 x + \log_3(x-2) = 1$

13 **a** Simplify: $\frac{1}{4}\log 81 + \log 12 - \log 4$

b If $x = \log_a 5$, write in terms of x: $\log_a(5a)$

c Write without logarithms: $\log_a N = 2\log_a d - \log_a c$

14 **a** If $A = \log_{10} P$, $B = \log_{10} Q$, and $C = \log_{10} R$, express in terms of A, B, and C: $\log_{10}\left(P^2 Q\sqrt{R}\right)$.

b Write $\frac{8}{\log_5 9}$ in the form $a\log_3 b$ where $a, b \in \mathbb{Z}$.

15 **a** How many terms are in the expansion of $\left(x + \frac{1}{x}\right)^5$?

b Use Pascal's triangle to help expand and simplify $\left(x + \frac{1}{x}\right)^5$.

16 Consider the binomial expansion of $(a+b)^6$.

a Write down the general term in the expansion.

b Given that $\binom{6}{4} = 15$, find the coefficient of a^4b^2.

17 Consider the expansion of $\left(3x - \frac{1}{x^2}\right)^4$.

a How many terms are in the expansion?

b Use the binomial coefficient $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to find the term in x.

SKILL BUILDER QUESTIONS (CALCULATORS)

1 Consider the arithmetic sequence 100, 130, 160, 190, ...

a Write an expression for the general term u_n.

b Find the first term in the sequence to exceed 1200.

c The sum of the first k terms of the sequence is 19 140. Find k.

2 Consider the geometric sequence with $u_5 = 18$ and $u_8 = 486$.

a Find the first term u_1 and common ratio r.

b Find the 12th term of the sequence.

c Find the sum of the first 10 terms of the corresponding geometric series.

3 Maria invested €800 on Jan 1st 2004.

a If her investment earns fixed interest of 7% per annum, what was it worth on Jan 1st 2012?

b How many years will it take for her investment to reach €4000?

4 Ying is training to run a marathon. In one week she ran 10 km on the first day and increased the distance by 10% on each subsequent day.

a How far did she run on the seventh day?

b What was the total distance she ran during the week?

5 Find the sum of the series:

a $10 + 14 + 18 + 22 + ... + 138$

b $6 - 12 + 24 - 48 + 96 - .... + 1536$

6 The sum of an infinite geometric series is 1.5, and its first term is 1. Find:

a the common ratio

b the sum of the first 7 terms, in rational form.

7 A sequence is defined by $u_n = 12\left(\frac{2}{3}\right)^{n-1}$.

a Prove that the sequence is geometric.

b Find the 5th term in rational form.

c Find: **i** $\sum_{n=1}^{\infty} u_n$

ii $\sum_{n=1}^{20} u_n$ correct to 4 decimal places.

8 Stan invests £3500 for 33 months at an interest rate of 8% p.a. compounded quarterly. Find its maturing value.

9 Consider the series $\sum_{k=1}^{\infty} 12(x-2)^{k-1}$.

a For what values of x will the series converge?

b Evaluate the sum of the series when $x = \sqrt{5}$.

10 Twins Pierre and Francesca were each given \$100 on their 15th birthday. They immediately put their money into their individual money boxes.

Each week throughout the next year they added a portion of their weekly pocket money. Pierre added \$10 each week. Francesca added 50 cents the first week, \$1 the next, \$1.50 the next, and so on, adding an extra 50 cents each subsequent week.

a Find the amount that each had added to his or her money box after 8 weeks.

b How much did Francesca add to her money box in the last week before her 16th birthday?

c Calculate the total amount they each had in their money boxes after one year.

11 Hayley and Patrick were training for a road cycling race. During the first week they both cycled 60 km. Hayley cycled an additional 20 km each subsequent week, whereas Patrick increased his distance by 20% each subsequent week.

a How far did each of them cycle in the 5th week of training?

b Who was the first to cycle 210 km in one week?

c What total distance did each cycle in the first 12 weeks?

12 Kapil invested 2000 rupees in a bank account on Jan 1st 2002. Each year thereafter, he invested another 2000 rupees into the same account. The account pays 8.25% per annum compounded annually.

a Find the total value of Kapil's investment immediately after he invested 2000 rupees on Jan 1st 2009.

b Would it have been a better option for Kapil to invest his money each year into an account paying 9% per annum simple interest? Justify your answer.

13 Paige has €500 to invest in an account that pays 7.2% p.a. compounded monthly. The formula $u_{n+1} = u_1 \times r^n$ can be used to model the investment, where n is the time in months.

a Explain why $r = 1.006$.

b How long will it take for Paige's investment to be worth €1000?

14 Solve for x:

a $5 \times 2^x = 160$ **b** $(1.25)^x = 10$ **c** $7e^x = 100$

15 **a** Solve for t: $200 \times e^{\frac{t}{4}} = 1500$

b Suppose $A = 125e^{-kt}$, and that $A = 200$ when $t = 3$. Find the value of k.

16 At the beginning of 2000 the number of koalas on an island was 2400. The numbers steadily increase each year according to $N(t) = 2400 + 250t$ where t is the number of years since 2000. The population of kangaroos on the same island is given by $K(t) = 3200 \times (0.85)^t$.

a What was the kangaroo population at the beginning of 2000?

b How many kangaroos were on the island at the beginning of 2005?

c How many koalas were on the island at the beginning of 2005?

d After how many years will the kangaroo population fall below 1000?

e When will the number of koalas exceed the number of kangaroos?

17 Consider the expansion of $\left(x + \frac{3}{x^2}\right)^9$.

a How many terms are in the expansion?

b Find the constant term.

18 Consider the expansion of $\left(x + 2y^3\right)^7$.

a Write down a formula for the general term.

b Find the coefficient of x^4y^9.

19 Consider the binomial expansion of $\left(2x - \frac{1}{x^2}\right)^{12}$. Find:

a the general term **b** the coefficient of x^3

c the constant term.

20 Consider the binomial expansion of $\left(kx + \frac{1}{\sqrt{x}}\right)^9$.

a Write down a formula for the general term.

b Given that the constant term is $-10\frac{1}{2}$, find k.

21 Find the coefficient of x^5 in the expansion of $(x+2)(1-x)^{10}$.

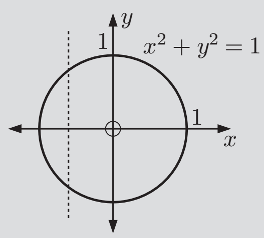
TOPIC 2: FUNCTIONS AND EQUATIONS

FUNCTIONS $f : x \mapsto f(x)$ OR $y = f(x)$

A **relation** is any set of points which connects two variables.

A **function** is a relation in which no two different ordered pairs have the same x-coordinate or first member. For each value of x there is only one value of y or $f(x)$. We sometimes refer to y or $f(x)$ as the **function value** or **image** of x.

We **test for functions** using the vertical line test. A graph is a function if no vertical line intersects the graph more than once.



For example, the graph of the circle $x^2 + y^2 = 1$ shows that this relation is not a function.

The **domain** of a relation is the set of values that x can take.