Chapter 9
MATHEMATICAL INDUCTION

EXERCISE 9A

1 The nth term of the sequence 3, 7, 11, 15, 19, ... is $4n-1$ for $n \in \mathbb{Z}^+$.

2 **a**
$3^1 = 3$ $\quad 1+2(1) = 3$
$3^2 = 9$ $\quad 1+2(2) = 5$
$3^3 = 27$ $\quad 1+2(3) = 7$
$3^4 = 81$ $\quad 1+2(4) = 9$

Our proposition is:
$3^n > 1+2n$ for $n = 2, 3, 4, 5, ...$
or for all $n \in \mathbb{Z}^+$, $n \geqslant 2$

b
$11^1 - 1 = 10$
$11^2 - 1 = 121 - 1 = 120$
$11^3 - 1 = 1331 - 1 = 1330$
$11^4 - 1 = 14\,641 - 1 = 14\,640$

Our proposition is:
$11^n - 1$ is divisible by 10 for all $n \in \mathbb{Z}^+$

c
$7^1 + 2 = 7 + 2 = 9 = 3 \times 3$
$7^2 + 2 = 49 + 2 = 51 = 3 \times 17$
$7^3 + 2 = 343 + 2 = 345 = 3 \times 115$
$7^4 + 2 = 2401 + 2 = 2403 = 3 \times 801$

Our proposition is:
$7^n + 2$ is divisible by 3 for all $n \in \mathbb{Z}^+$

d
$(1-\frac{1}{2}) = \frac{1}{2}$
$(1-\frac{1}{2})(1-\frac{1}{3}) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$
$(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$
$(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})(1-\frac{1}{5}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}$

Our proposition is: $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4}) .... \left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$ for all $n \in \mathbb{Z}^+$

3 **a**
$2 = 2 = 1 \times 2$
$2+4 = 6 = 2 \times 3$
$2+4+6 = 12 = 3 \times 4$
$2+4+6+8 = 20 = 4 \times 5$
$2+4+6+8+10 = 30 = 5 \times 6$

Our proposition is:
$2+4+6+8+10+ .... + 2n = n(n+1)$ for all $n \in \mathbb{Z}^+$
($2n$ ↑ nth term)

$\therefore \sum_{i=1}^{n} 2i = n(n+1)$ for all $n \in \mathbb{Z}^+$

b
$1! = 1$
$1! + 2 \times 2! = 1 + 2(2) = 5$
$1! + 2 \times 2! + 3 \times 3! = 1 + 4 + 18 = 23$
$1! + 2 \times 2! + 3 \times 3! + 4 \times 4! = 1 + 4 + 18 + 96 = 119$

where each number result is 1 less than a factorial number

$1 = 2! - 1$
$5 = 3! - 1$
$23 = 4! - 1$
$119 = 5! - 1$

Our proposition is:
$1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + ... + n \times n! = (n+1)! - 1$
for all $n \in \mathbb{Z}^+$

$\therefore \sum_{i=1}^{n} i \times i! = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$

c
$\frac{1}{2!} = \frac{1}{2} = \frac{2!-1}{2!}$

$\frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{5}{6} = \frac{3!-1}{3!}$

$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} = \frac{23}{24} = \frac{4!-1}{4!}$

$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} + \frac{4}{120} = \frac{119}{120} = \frac{5!-1}{5!}$

Our proposition is: $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\dfrac{4}{5!}+....+\dfrac{n}{(n+1)!}=\dfrac{(n+1)!-1}{(n+1)!}$ for all $n\in\mathbb{Z}^+$

$$\therefore\ \sum_{i=1}^{n}\frac{i}{(i+1)!}=\frac{(n+1)!-1}{(n+1)!} \quad \text{for all } n\in\mathbb{Z}^+$$

d $\dfrac{1}{2\times 5}=\dfrac{1}{10}$

$\dfrac{1}{2\times 5}+\dfrac{1}{5\times 8}=\dfrac{1}{10}+\dfrac{1}{40}=\dfrac{5}{40}=\dfrac{1}{8}=\dfrac{2}{16}$

$\dfrac{1}{2\times 5}+\dfrac{1}{5\times 8}+\dfrac{1}{8\times 11}=\dfrac{1}{10}+\dfrac{1}{40}+\dfrac{1}{88}=\dfrac{3}{22}$

$\dfrac{1}{2\times 5}+\dfrac{1}{5\times 8}+\dfrac{1}{8\times 11}+\dfrac{1}{11\times 14}=\dfrac{1}{7}=\dfrac{4}{28}$

10, 16, 22, 28 is arithmetic with $u_1=10,\ d=6$

$u_n=u_1+(n-1)d$
$=10+6(n-1)$
$=6n+4$

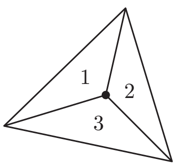
Our proposition is:

$\dfrac{1}{2\times 5}+\dfrac{1}{5\times 8}+\dfrac{1}{8\times 11}+\dfrac{1}{11\times 14}+....+\dfrac{1}{(3n-1)(3n+2)}=\dfrac{n}{6n+4}$ for all $n\in\mathbb{Z}^+$

{2, 5, 8, 11 are arithmetic with $u_1=2$, $d=3$ $\therefore$ $u_n=2+(n-1)3=3n-1$}

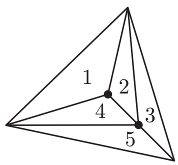
$$\therefore\ \sum_{i=1}^{n}\frac{1}{(3i-1)(3i+2)}=\frac{n}{6n+4} \quad \text{for all } n\in\mathbb{Z}^+$$

4 For $n=1$



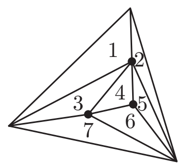
$T_1=3=2\times 1+1$

For $n=2$



$T_2=5=2\times 2+1$

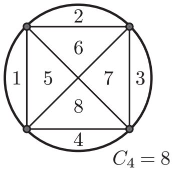
For $n=3$



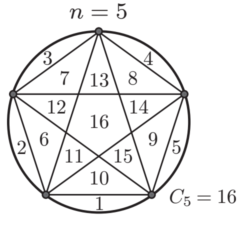
$T_3=7=2\times 3+1$

Our proposition is: The maximum number of triangles for n points within the original triangle is given by $T_n=2n+1$ for all $n\in\mathbb{Z}^+$.

5 **a** $n=4$



$n=5$

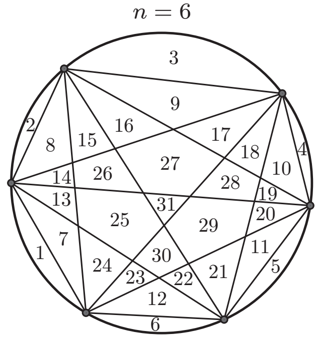


b When $n=1$, $C_1=1=2^0=2^{1-1}$
$n=2$, $C_2=2=2^1=2^{2-1}$
$n=3$, $C_3=4=2^2=2^{3-1}$
$n=4$, $C_4=8=2^3=2^{4-1}$
$n=5$, $C_5=16=2^4=2^{5-1}$

So, from the cases $n=1, 2, 3, 4, 5$, our conjecture is:

The number of regions for n points placed around a circle is given by $C_n=2^{n-1}$ for all $n\in\mathbb{Z}^+$.

c $n=6$



By the conjecture we expect $2^{6-1}=2^5=32$ regions, but there are only 31. So, we no longer believe the conjecture.

EXERCISE 9B.1

1 **a** If $n = 0$, $3^n + 1 = 3^0 + 1 = 2$ which is divisible by 2.

$3^n + 1 = (1+2)^n + 1$

$= 1^n + \binom{n}{1}2 + \binom{n}{2}2^2 + \binom{n}{3}2^3 + ... + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n + 1$

$= 2 + \binom{n}{1}2 + \binom{n}{2}2^2 + \binom{n}{3}2^3 + ... + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n$

$= 2\left(1 + \binom{n}{1} + \binom{n}{2}2 + \binom{n}{3}2^2 + ... + \binom{n}{n-1}2^{n-2} + \binom{n}{n}2^{n-1}\right)$

where the contents of the brackets is an integer.

$\therefore$ $3^n + 1$ is divisible by 2.

b P_n is: $3^n + 1$ is divisible by 2 for all integers $n \geqslant 0$.

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $3^0 + 1 = 2 = 1 \times 2$ $\therefore$ P_0 is true.

(2) If P_k is true, then $3^k + 1 = 2A$ where A is an integer, and $A \geqslant 1$.

Now $3^{k+1} + 1 = 3^1 3^k + 1$

$= 3(2A - 1) + 1$ {using P_k}

$= 6A - 3 + 1$

$= 6A - 2$

$= 2(3A - 1)$ where $3A - 1$ is an integer as $A \in \mathbb{Z}$

Thus $3^{k+1} + 1$ is divisible by 2 if $3^k + 1$ is divisible by 2.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all integers $n \geqslant 0$ {Principle of mathematical induction}

2 **a** If $n = 0$, $6^n - 1 = 6^0 - 1 = 0$ which is divisible by 5.

$6^n - 1 = (5+1)^n - 1$

$= 5^n + \binom{n}{1}5^{n-1} + \binom{n}{2}5^{n-2} + \binom{n}{3}5^{n-3} + ... + \binom{n}{n-1}5 + 1^n - 1$

$= 5^n + \binom{n}{1}5^{n-1} + \binom{n}{2}5^{n-2} + \binom{n}{3}5^{n-3} + ... + \binom{n}{n-1}5$

$= 5\left(5^{n-1} + \binom{n}{1}5^{n-2} + \binom{n}{2}5^{n-3} + \binom{n}{3}5^{n-4} + ... + \binom{n}{n-1}\right)$

where the contents of the brackets is an integer.

$\therefore$ $6^n - 1$ is divisible by 5.

b P_n is: $6^n - 1$ is divisible by 5 for all integers $n \geqslant 0$

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $6^0 - 1 = 0$ which is divisible by 5 $\therefore$ P_0 is true

(2) If P_k is true, then $6^k - 1 = 5A$ where $A \in \mathbb{N}$

Now $6^{k+1} - 1$

$= 6^1 6^k - 1$

$= 6(5A + 1) - 1$ {using P_k}

$= 30A + 6 - 1$

$= 30A + 5$

$= 5(6A + 1)$ where $6A + 1$ is an integer as $A \in \mathbb{N}$

Thus, $6^{k+1} - 1$ is divisible by 5 if $6^k - 1$ is divisible by 5.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all integers $n \geqslant 0$ {Principle of mathematical induction}

3 **a** P_n is: $n^3 + 2n$ is divisible by 3 for all positive integers n

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $1^3 + 2(1) = 3$ which is divisible by 3

(2) If P_k is true, then $k^3 + 2k = 3A$ where $A \in \mathbb{Z}$

Now $(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= (k^3 + 2k) + 3k^2 + 3k + 3$

$= 3A + 3k^2 + 3k + 3$ {using P_k}

$= 3(A + k^2 + k + 1)$ where $A + k^2 + k + 1$ is an integer as A and k are integers

Thus $(k+1)^3 + 2(k+1)$ is divisible by 3 if $k^3 + 2k$ is divisible by 3.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all positive integers n {Principle of mathematical induction}

b P_n is: $n(n^2 + 5)$ is divisible by 6 for all integers $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $1(1^2 + 5) = 1 \times 6 = 6$ which is divisible by 6 $\therefore$ P_1 is true

(2) If P_k is true, then $k(k^2 + 5) = 6A$ where A is an integer

Now $(k+1)[(k+1)^2 + 5] = (k+1)(k^2 + 2k + 1 + 5)$

$= (k+1)(k^2 + 2k + 6)$

$= k^3 + 2k^2 + 6k + k^2 + 2k + 6$

$= k^3 + 5k + [3k^2 + 3k + 6]$

$= k(k^2 + 5) + 3k(k+1) + 6$

$= 6A + 6 + 3k(k+1)$

We notice that $k(k+1)$ is the product of consecutive integers,
one of which must be even $\therefore$ $k(k+1) = 2B$ where $B \in \mathbb{Z}$

$\therefore$ $(k+1)[(k+1)^2 + 5] = 6A + 6 + 3(2B)$

$= 6(A + 1 + B)$ where $A + 1 + B \in \mathbb{Z}$

Thus $(k+1)[(k+1)^2 + 5]$ is divisible by 6 if $k(k^2 + 5)$ is divisible by 6.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all integers $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c P_n is: $7^n - 4^n - 3^n$ is divisible by 12 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $7^1 - 4^1 - 3^1 = 0$ which is divisible by 12 $\therefore$ P_1 is true

(2) If P_k is true, then $7^k - 4^k - 3^k = 12A$ where $A \in \mathbb{Z}$

Now $7^{k+1} - 4^{k+1} - 3^{k+1}$

$= 7(7^k) - 4(4^k) - 3(3^k)$

$= 7[12A + 4^k + 3^k] - 4(4^k) - 3(3^k)$ {using P_k}

$= 84A + 7(4^k) + 7(3^k) - 4(4^k) - 3(3^k)$

$= 84A + 3(4^k) + 4(3^k)$

$= 84A + 3 \times 4 \times 4^{k-1} + 4 \times 3 \times 3^{k-1}$

$= 12(7A + 4^{k-1} + 3^{k-1})$ where $k \geqslant 2$, $k \in \mathbb{Z}^+$

$= 12 \times$ an integer {as 4^{k-1} and 3^{k-1} are integers}

Thus $7^{k+1} - 4^{k+1} - 3^{k+1}$ is divisible by 12 if $7^k - 4^k - 3^k$ is divisible by 12.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

EXERCISE 9B.2

1 **a** P_n is: $1+2+3+4+....+n = \dfrac{n(n+1)}{2}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1$ and RHS $=\dfrac{1(2)}{2}=1$, $\therefore$ P_1 is true

(2) If P_k is true then $1+2+3+4+....+k = \dfrac{k(k+1)}{2}$

Thus $1+2+3+4+....+k+(k+1)$

$$= \frac{k(k+1)}{2}+k+1 \qquad \{\text{using } P_k\}$$

$$= \frac{k(k+1)}{2}+\frac{2(k+1)}{2} \qquad \{\text{to equalise denominators}\}$$

$$= \frac{(k+1)(k+2)}{2} \qquad \{\text{common factor of } \frac{(k+1)}{2}\}$$

$$= \frac{(k+1)([k+1]+1)}{2}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b P_n is: $1\times 2+2\times 3+3\times 4+....+n(n+1) = \dfrac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1\times 2=2$ and RHS $=\dfrac{1(2)(3)}{3}=2$, $\therefore$ P_1 is true

(2) If P_k is true then

$$1\times 2+2\times 3+3\times 4+....+k(k+1) = \frac{k(k+1)(k+2)}{3}$$

$\therefore$ $1\times 2+2\times 3+3\times 4+....+k(k+1)+(k+1)(k+2)$

$$= \frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \qquad \{\text{using } P_k\}$$

$$= \frac{k(k+1)(k+2)}{3}+\frac{3(k+1)(k+2)}{3} \qquad \{\text{to equalise denominators}\}$$

$$= \frac{(k+1)(k+2)(k+3)}{3} \qquad \{\text{common factor of } (k+1)(k+2)\}$$

$$= \frac{[k+1]([k+1]+1)([k+1]+2)}{3}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c P_n is: $3\times 5+6\times 6+9\times 7+12\times 8+....+3n(n+4) = \dfrac{n(n+1)(2n+13)}{2}$
for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=3\times 5=15$, RHS $=\dfrac{1\times 2\times(2+13)}{2}=15$, $\therefore$ P_1 is true

(2) If P_k is true, then

$$3\times 5+6\times 6+9\times 7+....+3k(k+4) = \frac{k(k+1)(2k+13)}{2}$$

Now $3\times 5+6\times 6+9\times 7+....+3k(k+4)+3(k+1)(k+5)$

$$= \frac{k(k+1)(2k+13)}{2}+3(k+1)(k+5) \quad \{\text{using } P_k\}$$

$$= \frac{k(k+1)(2k+13)}{2}+\frac{6(k+1)(k+5)}{2} \quad \{\text{to equalise denominators}\}$$

$$= \frac{(k+1)[k(2k+13)+6(k+5)]}{2} \quad \{\text{common factor}\}$$

$$= \frac{(k+1)[2k^2+19k+30]}{2}$$

$$= \frac{(k+1)(k+2)(2k+15)}{2}$$

$$= \frac{(k+1)([k+1]+1)(2[k+1]+13)}{2}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

d P_n is: $1^3+2^3+3^3+4^3+....+n^3=\dfrac{n^2(n+1)^2}{4}$ for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1^3=1$, RHS $=\dfrac{1^2(2)^2}{4}=1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1^3+2^3+3^3+....+k^3=\dfrac{k^2(k+1)^2}{4}$

Now $1^3+2^3+3^3+....+k^3+(k+1)^3$

$$= \frac{k^2(k+1)^2}{4}+(k+1)^3 \quad \{\text{using } P_k\}$$

$$= \frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4} \quad \{\text{equalising denominators}\}$$

$$= \frac{(k+1)^2[k^2+4(k+1)]}{4} \quad \{\text{common factor}\}$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

2 **a** The sum of the first n odd numbers $1+3+5+7+....+2n-1$ is the sum of the first n terms of an arithmetic series.
$u_1=1$, $d=2$. $\therefore$ $u_n=u_1+(n-1)d=1+2(n-1)=2n-1$.

Thus $S_n=\dfrac{n}{2}(2u_1+(n-1)d)$

$$= \frac{n}{2}(2\times 1+2(n-1))$$

$$= \frac{n}{2}(2+2n-2)$$

$$= \frac{n}{2}(2n)$$

$$= n^2$$

So, the sum of the first n odd numbers is n^2.