Mathematics: Core Topics HL

Chapter summaries

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CHAPTER 1: STRAIGHT LINES

A Lines in the Cartesian plane
B Graphing a straight line
C Perpendicular bisectors
D Simultaneous equations

Syllabus reference: SL 2.1, (AI)SL 3.5

We start the book with the study of straight lines. This should be a relatively gentle introduction for most students as they start the school year, and should be worked through quickly. Perpendicular bisectors is included in this common chapter, even though it is only listed in the Applications and Interpretation syllabus, as it is a very sensible application of straight lines, and is something the Analysis and Approaches students would benefit from being familiar with.

The syllabus defines the general form equation of a line as $ax + by + d = 0$. However, we prefer to use the form $ax + by = d$. There are several reasons for this. For example, it is easier to find the intercepts of a line written in this form, and it is more logical to express equations in problem solving this way (for example, 2 bats and 5 balls cost \$30 $\Rightarrow 2x + 5y = 30$).

Therefore, we predominantly use the form $ax + by = d$, but we mention that, in an exam, students may be asked to write their answer in the form $ax + by + d = 0$. In several exercise questions, we also ask students to write their answer in the form $ax + by + d = 0$.

The solution of simultaneous linear equations provides a good platform for solving 3×3 systems of equations later in the course. The algebraic solution of simultaneous equations is also used in other contexts throughout the course, such as in arithmetic sequences.

CHAPTER 2: SETS AND VENN DIAGRAMS

A Sets
B Intersection and union
C Complement of a set
D Special number sets
E Interval notation
F Venn diagrams
G Venn diagram regions
H Problem solving with Venn diagrams

This chapter gives students access to much of the notation associated with probability (complement, intersection and union), as well as an introduction to Venn diagrams. This will allow students to focus on its application to probability once it is encountered again in Chapter 11. Students are also introduced to notation for special number sets that will be used throughout the book. The work on interval notation here will allow students to familiarise themselves with this notation before it is used in, for example, the domain and range of functions.

If time constraints are a factor for classes, this is a chapter which could potentially be skipped, or moved through very quickly.

When discussing the union of two sets, we are careful not to describe the union of A and B as the elements that are "in A or B or both". We feel that this wording gives the implication that, if the "or both" were excluded, we would only be talking about the exclusive union of A and B.

CHAPTER 3: SURDS AND EXPONENTS

A Surds and other radicals
B Division by surds
C Exponents
D Laws of exponents
E Scientific notation

Syllabus reference: SL 1.1, SL 1.5

Classes should work through this chapter quickly if the students are already familiar with this content.

When studying laws of exponents, students should be able to distinguish between laws which are a direct result of the original definition of exponents (such as $a^m \times a^n = a^{m+n}$), and those which have been defined in a certain way in order to be consistent with the existing laws (such as $a^0 = 1$).

Writing expressions such as $\frac{4}{x} - \frac{5}{x^3}$ with negative indices is practised, as it is an important skill to develop before students encounter calculus.

The chapter ends with a Discussion about how very large values could be evaluated. Students should consider other forms of technology, as well as things like cancelling common factors that exist in the numerator and denominator, and the order in which the evaluation is performed. For example, do you need to completely evaluate the numerator first, and then divide through by the denominator?

Rational indices are only required for Analysis and Approaches students, so it appears at the start of the Exponential functions chapter in the Mathematics: Analysis and Approaches HL book.

CHAPTER 4: EQUATIONS

A Power equations
B Equations in factored form
C Quadratic equations
D Solving polynomial equations using technology
E Solving other equations using technology

Syllabus reference: SL 2.4, (AA)SL 2.7, (AA)SL 2.10, (AI)SL 1.8, (AA)AHL 2.12

This chapter covers quadratic equations, power equations of the form $x^n = k$, and solving equations by technology. Students will benefit from studying this material early in the course, as solving these types of equations comes up in many topics, such as sequences.

In Section C, Subsections C.4 (The discriminant of a quadratic) and C.5 (The sum and product of the roots) are for Analysis and Approaches students only. These students should find Subsection C.5 a good introduction for finding the sum and product of roots for general polynomials later in the course.

We have introduced the notion of "zeros" here, even though they are more appropriately applied to expressions or functions, as this is where they are referenced in the syllabus.

In the Discussion at the end of Section E, regarding the merits of using the solver versus graphical methods on the calculator to solve equations, students should talk about the work required to rearrange the equation into a suitable form to use the solver, as well as the potential to miss solutions when using the graphical method due to an inappropriate viewing window.

This chapter also details how to solve quadratic equations using technology, which is only for the Applications and Interpretation students.

CHAPTER 5: SEQUENCES AND SERIES

A Number sequences
B Arithmetic sequences
C Geometric sequences
D Growth and decay
E Financial mathematics
F Series
G Arithmetic series
H Finite geometric series
I Infinite geometric series

Syllabus reference: SL 1.2, SL 1.3, SL 1.4, (AA)SL 1.8, (AI)AHL 1.11

In Section B.2, we consider real-life examples which can be modelled approximately, but not exactly, by an arithmetic sequence. We therefore use an exact arithmetic sequence as an *approximation* of the real-world quantity.

When finding the general term of a geometric sequence given two of its terms, students should be aware that there will be two possible answers if the difference between the given terms is even. This is where solving equations of the form $x^n = k$ in the previous chapter will prove valuable.

When using sequences in contexts such as compound interest and depreciation, we define the initial conditions with the "zeroth" term u_0. This allows us to make more sensible deductions about the context. For example, the value of the investment after 4 years is u_4.

The simpler finance questions in this chapter are to be solved using formulae based on that for geometric sequences. More complex problems will be solved using technology. Using technology to solve finance problems will be expanded upon in the Mathematics: Applications and Interpretation HL book.

CHAPTER 6: MEASUREMENT

A Circles, arcs, and sectors
B Surface area
C Volume
D Capacity

Syllabus reference: SL 3.1, SL 3.4

Much of the material in this chapter will be familiar to many students, and classes should skip through it quickly if this is the case. In this chapter we have attempted to provide some questions about measurement that students may not be as familiar with, such as finding the radius of a cylinder given its volume and height.

We have also included some investigations detailing the derivation of some surface area and volume formulae, which use the sigma notation developed in the previous chapter. These investigations should provide an interesting challenge for the more able student. In particular, it is hoped that the derivation of the surface area of a sphere at the end of Investigation 2 should spark some interesting class discussion.

The chapter contains an investigation on density, which students can complete if they wish, but can be skipped if need be.

CHAPTER 7: RIGHT ANGLED TRIANGLE TRIGONOMETRY

A Trigonometric ratios
B Inverse trigonometric ratios
C Right angles in geometric figures
D Problem solving with trigonometry
E True bearings
F The angle between a line and a plane

Syllabus reference: SL 3.1, SL 3.2, SL 3.3

In this chapter, students will build on the work done in the previous chapter, using trigonometry to find areas of shapes, and volumes of solids. The early sections of this chapter should be worked through quickly if students are comfortable with the content. The last question of Section D provides an interesting application of trigonometry, involving measuring the parallax of the star 61 Cygni. Teachers should use questions like this as an opportunity to promote the historical interest of the work the students are doing.

The final section on finding an angle between a line and a plane may present a challenge for some students. This work will provide the platform for using trigonometry in 3-dimensional space in Chapter 10.

CHAPTER 8: THE UNIT CIRCLE AND RADIAN MEASURE

- **A** Radian measure
- **B** Arc length and sector area
- **C** The unit circle
- **D** Multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$
- **E** The Pythagorean identity
- **F** Finding angles
- **G** The equation of a straight line

Syllabus reference: (AA)SL 3.4, (AA)SL 3.5, (AA)SL 3.6, (AI)AHL 3.7, (AI)AHL 3.8

We start this chapter with an introduction to radian measure. The arc length and sector area have already been covered in Chapter 6, but with the angle given in degrees. We now present the much simpler formulae when the angle is given in radians.

In this chapter we consider the complete unit circle, allowing the students to give meaning to the trigonometric ratios for *any* angle. Students should be encouraged to become familiar with the identities involving supplementary, complementary, and negative angles, as an understanding of these will help with the formulation of trigonometric functions in Chapter 17.

Section C contains a Discussion about identities. Students should be able to distinguish between equations, which are true for only particular values of a variable, and identities, which are true for *all* values of the variable. Examples of identities include the Pythagorean identity, as well as expansion and factorisation identities such as $(a+b)(a-b) = a^2 - b^2$.

In Section F, we use the inverse trigonometric ratios, as well as the properties of supplementary and negative angles, to find angles with a given trigonometric ratio, over a particular interval. Mastering this work requires a good understanding of the angle the calculator gives when calculating an inverse trigonometric ratio, as well as the ability to find other angles with the same trigonometric ratio. This work will provide a solid grounding for solving trigonometric equations later in the course.

Section G (The equation of a straight line) is for Analysis and Approaches students only.

CHAPTER 9: NON-RIGHT ANGLED TRIANGLE TRIGONOMETRY

- **A** The area of a triangle
- **B** The cosine rule
- **C** The sine rule
- **D** Problem solving with trigonometry

Syllabus reference: SL 3.2, SL 3.3, (AA)SL 3.5, (AI)AHL 3.8

It is likely that this is not the first time students have studied non-right angled triangle trigonometry, so it is important that classes should not linger on this chapter if the students are comfortable with the content. Because students should have seen this material before, we have included some challenging questions and activities within the chapter to push the more able students.

The ambiguous case of the sine rule is included in the syllabus of both HL courses. Students are led through an investigation in which they are given two sides and a non-included angle of a triangle. They should find that this information does not always produce a unique answer, so there may be more than one possible answer when finding angles using the sine rule.

CHAPTER 10: POINTS IN SPACE

- **A** Points in space
- **B** Measurement
- **C** Trigonometry

Syllabus reference: SL 3.1

In this chapter we extend the coordinate system to consider points and lines in 3-dimensional space. This work may be challenging for students who struggle with 3D spatial awareness.

Many of the techniques involving calculating volumes and surface areas of solids, and the calculation of angles, were established in Chapters 6, 7, and 9. However, instead of being given the side lengths of figures, students will be given the coordinates of the vertices, and must first calculate the relevant distances.

In the Discussion at the end of Section B, students must describe an algebraic test to determine whether a point (x, y, z) lies inside a selection of 3-dimensional solids. For a simple case such as the sphere, it should be clear that the point must be within 5 units of the origin, so its coordinates must satisfy $x^2 + y^2 + z^2 \leqslant 25$. For more complicated objects such as the cone, students should recognise that, on the X-Y plane, the point must lie within a circle, whose radius depends on the z-coordinate. For z-coordinate z, the circle has radius $\frac{1}{3} \times (6 - z)$, so the coordinates of the point must satisfy $x^2 + y^2 \leqslant \left(\frac{1}{3} \times (6 - z)\right)^2$, where $0 \leqslant z \leqslant 6$.

CHAPTER 11: PROBABILITY

- **A** Experimental probability
- **B** Two-way tables
- **C** Sample space and events
- **D** Theoretical probability
- **E** Making predictions using probability
- **F** The addition law of probability
- **G** Independent events
- **H** Dependent events
- **I** Conditional probability
- **J** Formal definition of independence
- **K** Bayes' theorem

Syllabus reference: SL 4.5, SL 4.6, (AA)SL 4.11, (AA)AHL 4.13

From Section C onwards, there is a strong emphasis on the concepts studied in the Sets and Venn diagrams chapter. Because of this, we have put the addition law of probability *before* independent/dependent events to reinforce the use of Venn diagrams in probability.

In previous books, we had dedicated sections for tree diagrams and 2-dimensional grids. However, doing this disrupted the flow of the chapter as it meant reintroducing the same concepts (for example, sample space and theoretical probability) multiple times in slightly different ways. Instead, tree diagrams and 2-dimensional grids are spread throughout the entire chapter and their usage is generally introduced via worked examples.

We have kept Section E (Making predictions using probability) in the Probability chapter rather than putting it with expectation for discrete random variables. It is important to make a distinction between "the number of times we expect an event to occur out of many trials" and the "expected/average outcome of one trial of an experiment". Thus we have avoided the use of the word "expectation" and its variations in this section as the latter concept will be dealt with in the chapter on discrete random variables. The Discussion in this chapter should lead students to understand that the number of times we *expect* an event to occur may not be an integer, and this is not a problem, as this value is an indication of what we expect to occur *on average* in the long-term, rather than what will actually happen in a particular instance.

Sections J (Formal definition of independence) and K (Bayes' theorem) are intended for students doing the Mathematics: Analysis and Approaches HL course. It is highly recommended that students who are undecided on which course they want to do should work through this section as well.

Some of the Review set questions were written to be answered using Bayes' theorem. However, these questions can be answered without explicitly mentioning Bayes' theorem by using a tree diagram instead. So these questions have not been marked as "Analysis only" in the Review sets.

CHAPTER 12: SAMPLING AND DATA

A Errors in sampling and data collection
B Sampling methods
C Writing surveys
D Types of data
E Simple discrete data
F Grouped discrete data
G Continuous data

Syllabus reference: SL 4.1, SL 4.2, (AI)AHL 4.12

In this chapter we focus on data collection, organisation, and the display of data. These are the first three steps in a statistical investigation.

In the Discussion at the end of Section A, students should find that companies offer incentives for completing surveys to encourage more people to do so, thus providing a larger sample size and more meaningful results. However, students should also ponder whether this will affect the integrity of the results, as people may complete the survey as quickly as possible, possibly untruthfully, just so they can receive the incentive.

The Discussion at the end of Section B invites students to consider the merits of the "Brexit" referendum of 2016. In their discussions, students should recognise that the referendum was voluntary, and consider the potential for bias in the responses which this brings.

Section C (Writing surveys) is only intended for students doing the Mathematics: Applications and Approaches HL course. This section looks at the actual data collection in the context of surveys and questionnaires. As a result, there is a larger focus on *language* and students who struggle with English may find this section difficult. Beyond the exercise questions, a practical activity on survey writing is probably the best way for students to practice and apply these skills.

We have decided to put "precise questioning" in as a research exercise as we feel that it has little relevance to statistics. Although the steps involved in precise questioning may resemble the steps in a statistical investigation, the underlying goals are quite different as it is usually treated as a business tool.

CHAPTER 13: STATISTICS

A Measuring the centre of data
B Choosing the appropriate measure
C Using frequency tables
D Grouped data
E Measuring the spread of data
F Box and whisker diagrams
G Outliers
H Parallel box and whisker diagrams
I Cumulative frequency graphs
J Variance and standard deviation

Syllabus reference: SL 4.2, SL 4.3

At the end of Section E, there is a Discussion regarding the different interpretations of the interquartile range from different sources of technology. It is hoped that students conclude that it is important to understand what the quartiles Q_1 and Q_3 represent, however the exact method for computing these values is not as important, as the values are simply an estimate anyway. The interpretation of these values will not change.

In Section J, we briefly mention the sample formulae for variance and standard deviation as justification for the multiple values that most calculator models show when calculating summary statistics for the Mathematics: Analysis and Approaches HL course.

However, for the Mathematics: Applications and Interpretation HL course, it is necessary that these students understand that the sample variance formula $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$ is an unbiased estimate of the population variance σ^2. This will become important later in the course when estimators are considered.

In any exercise questions where the variance or standard deviation is asked for, we give both the poulation *and* sample values in the answers. Teachers should give guidance as to which value students should give based on the course they are doing.

CHAPTER 14: QUADRATIC FUNCTIONS

- **A** Quadratic functions
- **B** Graphs of quadratic functions
- **C** Using the discriminant
- **D** Finding a quadratic from its graph
- **E** The intersection of graphs
- **F** Problem solving with quadratics
- **G** Optimisation with quadratics
- **H** Quadratic inequalities

Syllabus reference: SL 2.4, (AA)SL 2.6, (AA)SL 2.7, (AI)SL 2.5

This chapter follows from the work done in Chapter 4 involving quadratic equations.

We place this chapter ahead of the Functions chapter, so that when we get to the Functions chapter we can use quadratic functions as tools for studying domain, range, and composite and inverse functions.

Section C (Using the discriminant) is more relevant to Analysis and Approaches HL students. Applications and Interpretation HL students could potentially skip this section, as well as the corresponding questions in Sections E and H.

In Section E, students solve quadratic inequalities graphically by identifying where one graph lies above or below another. An algebraic approach to quadratic inequalities is given in Section H. In this section, sign diagrams are introduced as tools to help solve quadratic inequalities. At this stage, they are taught without mention of points where the function is undefined, as this is not relevant for quadratics. The treatment of sign diagrams is completed in Chapter 15, where we draw sign diagrams for rational functions, and consider points at which the function is undefined.

CHAPTER 15: FUNCTIONS

- **A** Relations and functions
- **B** Function notation
- **C** Domain and range
- **D** Rational functions
- **E** Composite functions
- **F** Inverse functions

Syllabus reference: SL 2.2, SL 2.3, SL 2.4, (AA)SL 2.5, (AA)SL 2.8, (AI)AHL 2.7

Rather than having a section specifically devoted to graphing functions using technology, this work has been absorbed into the other sections of this chapter. For example, in Section C, students must use technology to find the turning points of a function, as well as the value of the function at the endpoints of the domain, to help determine the range of the function.

Although rational functions are not explicitly mentioned in the Applications and Interpretation HL syllabus, the study of reciprocal functions will be useful ahead of studying inverse variation later in the course. It will also introduce students to vertical and horizontal asymptotes. This is where students learn how to indicate where a function is undefined on a sign diagram. Students should be encouraged to construct sign diagrams for rational functions, as they are helpful for determining the behaviour of the function near its asymptotes.

In Section F, students should understand that a function may not be one-to-one over its natural domain, but if we restrict the domain of the function such that it is one-to-one, we can find an inverse of that function. Students should understand the effect of the domain restriction when finding the form of the inverse function, as well as the domain and range of the inverse function.

CHAPTER 16: TRANSFORMATIONS OF FUNCTIONS

A Translations
B Stretches
C Reflections
D Miscellaneous transformations
E The graph of $y = \frac{1}{f(x)}$

Syllabus reference: (AA)SL 2.11, (AI)AHL 2.8

In this chapter we build on the function work done in Chapter 15 to consider transformations of functions. It is emphasised that the scale factors for the vertical and horizontal stretches must be positive: we should not consider the transformation from $y = x^2$ to $y = -2x^2$ as a vertical stretch with scale factor -2. Instead, it should be considered as a vertical stretch with scale factor 2, followed by a reflection in the x-axis. This allows us to preserve the uniqueness of tranformations.

In the Discussion in Section B, students should conclude that for $y = x^2$, the transformation by a *vertical* stretch with scale factor 4 is equivalent to a *horizontal* stretch with scale factor $\frac{1}{2}$. However, not every vertical stretch has an equivalent horizontal stretch in this manner. For example, consider a function which does not pass through the origin. A vertical stretch will change the y-intercept of the function, but a horizontal stretch cannot change the y-intercept.

We use the term "stretch" as given in the syllabus, but we also use the term "dilation", as it is a more general term which does not carry any particular geometric connotations. For example, it may be misleading for students to consider a vertical stretch with scale factor $\frac{1}{2}$, as this is a compression rather than a stretch.

Composite transformations are introduced as additional transformations are discovered, rather than as a section of its own.

In the Discussion at the end of Section C, students are asked for which combinations of transformations is the order in which the transformations are performed important. To get students started on this, it could be illustrated to them that if a function is to be reflected in the x-axis and the y-axis, the order in which these occur do not matter. However, if a function is to be reflected in the x-axis and then rotated, the order in which these occurs does matter, as you will obtain different results. As students dig deeper into this, it may be useful to consider the coordinates that are affected by the transformations. If one transformation only affects the x-coordinate, and one only affects the y-coordinate, then the transformations act *independently* of each other, and the order in which they are performed does not matter.

Section E (The graph of $y = \frac{1}{f(x)}$) is most relevant for Analysis and Approaches students. It is here because it will be useful for graphing reciprocal trigonometric functions in Chapter 1 of the Mathematics: Analysis and Approaches HL book. Applications and Interpretation students may skip this section if they wish.

CHAPTER 17: TRIGONOMETRIC FUNCTIONS

A Periodic behaviour
B The sine and cosine functions
C General sine and cosine functions
D Modelling periodic behaviour
E Fitting trigonometric models to data
F The tangent function
G Trigonometric equations
H Using trigonometric models

Syllabus reference: (AA)SL 3.7, (AI)SL 2.5, (AI)AHL 2.9, (AI)AHL 3.8

Here we use the transformations studied in Chapter 16 to construct graphs of trigonometric functions. We use the general form $y = a \times \sin(b(x - c)) + d$ rather than $y = a \times \sin(b(x + c)) + d$ as given in the Mathematics: Analysis and Approaches syllabus, so that we can talk sensibly about a horizontal translation of c units and a vertical translation of d units. This is consistent with the transformation $f(x - a)$ described in the Functions section of the syllabus.

Section E (Fitting trigonometric models to data) is for Analysis and Approaches students only. In this section students are given data points which display periodic behaviour, and must find a model to fit the data. Students can use technology to check their models, but they should be made aware that the calculator may not give the model in the form $y = a \times \sin(b(x - c)) + d$, and the value of c may be different to the one students obtained. However, students should be able to use the trigonometric identities to confirm that their answer and the answer given by their calculator are equivalent.

In Section G, students must solve trigonometric equations graphically and algebraically. Applications and Interpretation students must complete G.1 and G.2 involving solutions from graphs and technology, but may leave some of the more complicated questions in G.3 if need be.